

STAT 902 Jan 5 2009 Andrew.

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- click on Chapter 01 - Preliminaries

§ 1.2.1 of Notes

Let E be a set.

Let $\mathcal{P}(E)$ denote the collection of all possible subset of E including \emptyset . $\Phi \subseteq \mathcal{P}(E)$

Let \mathcal{C} be a given collection of subsets of E .

i.e. $\mathcal{C} \subseteq \mathcal{P}(E)$

(i) \mathcal{C} is a π -class including $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$

e.g. $E = \mathbb{R}$, $\mathcal{C} \stackrel{\Delta}{=} \{(-\infty, 1], (0, \infty)\}$ NOT a π -class

$E = \mathbb{R}$, $\mathcal{C} \stackrel{\Delta}{=} \{(-\infty, a] : a \in \mathbb{R}\}$ is a π -class.



(ii) \mathcal{C} is an algebra when \mathcal{C} is a π -class with the further properties that (i) $E \in \mathcal{C}$ (ii) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$

(iii) \mathcal{C} is a σ -algebra with the following further properties:

for any sequence $\{A_m, m=1, 2, \dots\}$ of members of \mathcal{C} (i.e.

$A_m \in \mathcal{C}$) we have

$$\bigcap_{m=1}^{\infty} A_m \in \mathcal{C}$$

If E is a set and \mathcal{S} is a σ -alg. of subsets of E , then

(E, \mathcal{S}) is called a measurable space.

Let E be a given set. Fix some $\mathcal{C} \subseteq \mathcal{P}(E)$

Let $\mathcal{L} = \{\mathcal{F} \subseteq \mathcal{P}(E), \mathcal{F}$ is a σ -alg. $\mathcal{C} \subseteq \mathcal{F}\}$

e.g. $\mathcal{P}(E) \in \mathcal{L}$

Topic 7

SOP TAKE

Put $\mathcal{G}^* = \{A \subseteq E : A \in \mathcal{G} \text{ for each } \mathcal{G} \subseteq \mathcal{E}\}$

Then: i.e. $\mathcal{G}^* = \bigcap_{\mathcal{G} \subseteq \mathcal{E}} \mathcal{G}$

Then (i) \mathcal{G}^* is a σ -algebra over E .

(ii) $\mathcal{C} \subseteq \mathcal{G}^* \subseteq \mathcal{E} \subseteq \mathcal{G}^*$

(iii) For each $\mathcal{G} \subseteq \mathcal{E}$, we have $\mathcal{G}^* \subseteq \mathcal{G}$

In short, \mathcal{G}^* is the smallest σ -alg over E which is large enough to include all elements of \mathcal{C} , call \mathcal{G}^* the σ -algebra generated by \mathcal{C} .

Denote \mathcal{G}^* by $\sigma(\mathcal{C})$

Examples:

(a) $E = \mathbb{R}$. $\mathcal{C} = \{(-\infty, a] : a \in \mathbb{R}\}$

call $\sigma(\mathcal{C})$ the Borel σ -alg on the real line, and denote it by $\mathcal{B}(\mathbb{R})$

(clear that $\mathcal{B}(\mathbb{R}) \subseteq \sigma(\mathcal{C})$)

Borel thought that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$

In 1910, it was shown by Vitali (Ztahian) that

$\mathcal{B}(\mathbb{R}) \neq \sigma(\mathcal{C})$

using axiom of choice

(b) $E = \mathbb{R}^d$

for each $a \in \mathbb{R}^d$,

$$(-\infty, a] = \{x \in \mathbb{R}^d : x^i \leq a^i, i=1, \dots, d\}$$

$$\mathcal{C} = \{(-\infty, a] : a \in \mathbb{R}^d\}$$

semi-infinite cells

(a) $\sigma(\mathcal{C})$ the Borel σ -alg on \mathbb{R}^d , denote it by $\mathcal{B}(\mathbb{R}^d)$

(c) $E = \overline{\mathbb{R}}$ extended real line

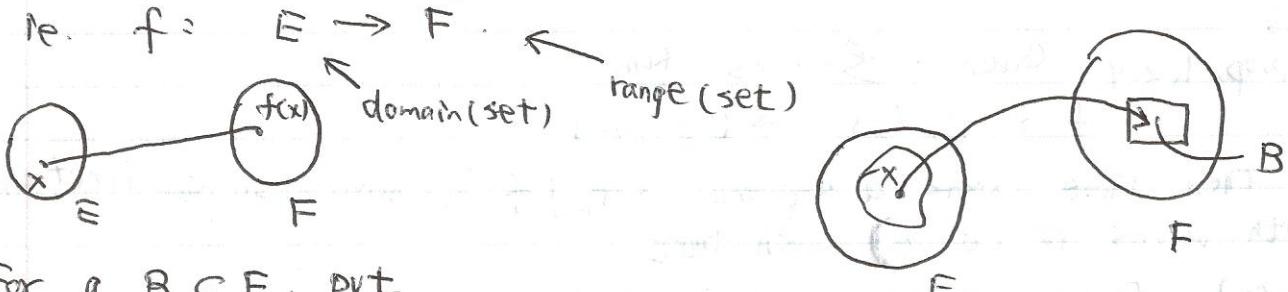
$$\stackrel{\Delta}{=} \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$$

$$\mathcal{C} = \{(-\infty, \infty], (-\infty, a], \{+\infty\}, \{-\infty\} : a \in \mathbb{R}\}$$

call $\mathcal{C}(\mathbb{R})$ the Borel σ -alg. on \mathbb{R} , denote by $\mathcal{B}(\mathbb{R})$
 § 1.2.3. Let E and F be sets.

and let f be a mapping from E into F

$$\text{i.e. } f: E \rightarrow F$$



for a $B \subset F$, put

$$f^{-1}[B] \triangleq \{x \in E : f(x) \in B\}$$

"Inverse image of B under f "

Take measurable spaces (E, \mathcal{S}) and (F, \mathcal{B}) . Then a mapping

$$f: E \rightarrow F$$
 is said to be

\mathcal{S} \mathcal{B} -meas. when $f[B] \in \mathcal{S}$ for each $B \in \mathcal{B}$

when $F = \mathbb{R}$, we always take $\mathcal{B} = \mathcal{B}(\mathbb{R})$

$$F = \mathbb{R}, \quad \mathcal{B} = \mathcal{B}(\mathbb{R})$$

$$F = \mathbb{R}^d, \quad \mathcal{B} = \mathcal{B}(\mathbb{R}^d)$$

when $A \subset E$, define $I_A: E \rightarrow \mathbb{R}$ by

$$I_A(x) \triangleq \begin{cases} 1 & \text{when } x \in A \\ 0 & \text{when } x \notin A \end{cases}$$

"Indicator function of A ".

When (E, \mathcal{S}) is a meas. space and $A \subset E$, then

$$I_A \text{ is } \mathcal{S}\text{-meas iff. } A \in \mathcal{S}$$

A simple function on a (measurable space (E, \mathcal{S})) is an \mathbb{R} -valued function

which is a linear combination of indicator functions on E .

i.e. $f(x) = \sum_{i=1}^n x_i I_{A_i}(x)$ for finitely many $x_i \in \mathbb{R}$, $i = 1, \dots, n$.

$$A_i \subset E, \quad i = 1, \dots, n$$

Fix a σ -alg. \mathcal{S} on E .

Then (elementary exercise.)

A simple function of the form $(*)$ is \mathcal{S} -meas. iff.

each $A_i \in \mathcal{S}, i=1, \dots, n$.

§

Prop. 1.2.9 Given a \mathcal{S} -meas. func.

$$f : (E, \mathcal{S}) \rightarrow [0, \infty]$$

Then there exists a sequence $\{f_n\}$ of \mathcal{S} -meas. simple functions with values in $[0, \infty)$ such that

$$(a) 0 \leq f_n(x) \leq f_{n+1}(x) \text{ for } x \in E, n=1, 2, \dots$$

$$(b) \lim_{n \rightarrow \infty} f_n(x) = f(x), \text{ for } x \in E.$$

we are given a set E a meas. space (F, \mathcal{B})

a mapping $f : E \rightarrow (F, \mathcal{B})$

Define $\sigma\{f\} \triangleq \{ f^{-1}[B] : B \in \mathcal{B} \}$

OBS. $\sigma\{f\} \subset \mathcal{P}(E)$

$$\text{Recall. } f^{-1}\left(\bigcap_{n=1}^{\infty} B_n\right) = \bigcap_{n=1}^{\infty} f^{-1}[B_n]$$

From this we can see

$\sigma\{f\}$ is a σ -alg. over E .

Moreover, it is immediate that

$$f^{-1}[B] \in \sigma\{f\} \text{ for each } B \in \mathcal{B}.$$

Thus. f is $\sigma\{f\} / \mathcal{B}$ -meas.

Finally, for any σ -alg. \mathcal{S} over E st. \mathcal{S} is $\mathcal{S} / \mathcal{B}$ -meas.

we have

$$\sigma\{f\} \subset \mathcal{S}$$

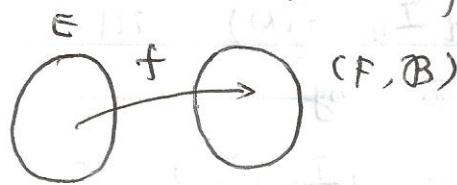
i.e. $\sigma\{f\}$ is the smallest algebra on E which makes f measurable.

Doob Jan 9th, 2009

MC4064 probability theory II

5-

Given a set E
meas. space (F, \mathcal{B})
 $f: E \rightarrow (F, \mathcal{B})$



$$\sigma\{f\} = \{f^{-1}(B): B \in \mathcal{B}\}$$

σ -algebra over E .

f is $\sigma\{f\}/\mathcal{B}$ -meas.

If \mathcal{S} is any σ -alg on E st. f is \mathcal{S}/\mathcal{B} -meas. then
 $\sigma\{f\} \subset \mathcal{S}$

Theorem 1.3.1. [Doob] Given a set E , a meas. space (F, \mathcal{B}) and a function $f: E \rightarrow (F, \mathcal{B})$

$(E, \sigma\{f\})$ (F, \mathcal{B}) . If $Z: E \rightarrow \bar{\mathbb{R}}$
is $\sigma\{f\}$ -meas. Then there exists
a \mathcal{B} -meas function $\psi: F \rightarrow \bar{\mathbb{R}}$
such that $Z(x) = \psi(f(x))$ all $x \in E$.
ie. Z is a function of the function f .

Interpretation:

Suppose that f is a rv. giving the price of IBM stock at close of trade tomorrow afternoon.

Suppose Z is the price of another asset also at close of trade tomorrow afternoon.

If Z is $\sigma\{f\}$ -meas. then its value is decided entirely by f .
(the price of IBM stock)

Proof: (i) Suppose Z is a $\sigma\{f\}$ -meas. simple fn.

$$\text{ie. } Z(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x) \text{ for all } x \in E. \quad \text{--- (1)}$$

where $\alpha_i \in \mathbb{R}$. $A_i \in \sigma\{f\}$, $i=1, \dots, n$.

But $A_i = f^{-1}(B_i)$ for some $B_i \in \mathcal{B}$

$$\text{But } I_{A_i}(x) = I_{f^{-1}(B_i)}(x) = I_{B_i}(f(x)) \text{ for all } x \in E \quad \text{--- (2)}$$

↑ easy exercise

From ① ②

$$z(x) = \sum_{i=1}^n \alpha_i I_{B_i}(f(x)) \quad \text{all } x \in E \quad \dots \textcircled{3}$$

Define $\psi : F \rightarrow \bar{\mathbb{R}}$ by

$$\psi(\xi) \triangleq \sum_{i=1}^n \alpha_i I_{B_i}(\xi) \quad \xi \in F \quad \dots \textcircled{4}$$

$\uparrow \in \mathcal{B}$

i.e. ψ is \mathcal{B} -meas.

From ③ ④. $z(x) = \psi(f(x))$ all $x \in E$

(ii) suppose z is $\sigma\{f\}$ -meas. and $[0, \infty]$ -valued

By prop 1.2.9 there exists a sequence $z_n : F \rightarrow \bar{\mathbb{R}}$ of

$\sigma\{f\}$ -meas. simple functions st.

$$\lim_n z_n(x) = z(x) \quad \text{for each } x \in E \quad \dots \textcircled{5}$$

By (i) $\exists \mathcal{B}$ -meas. $\psi_n : F \rightarrow \bar{\mathbb{R}}$ st.

$$z_n(x) = \psi_n(f(x)) \quad \text{all } x \in E \quad \dots \textcircled{6}$$

Define $C \triangleq \{\xi \in F : \liminf_n \psi_n(\xi) = \limsup_n \psi_n(\xi)\} \quad \dots \textcircled{7}$

For each $x \in E$, we have

from ⑤, ⑥ that

$$\lim_n \underbrace{\psi_n(f(x))}_{z_n(x)} = z(x)$$

$$\text{i.e. } \liminf_n \psi_n(f(x)) = \limsup_n \psi_n(f(x))$$

i.e. $f(x) \in C$.

i.e. $f(x) \in C$, all $x \in E \quad \dots \textcircled{8}$

since each ψ_n is \mathcal{B} -meas. we also have

$$C \in \mathcal{B} \quad \dots \textcircled{9}$$

Define $\psi : F \rightarrow \bar{\mathbb{R}}$ by

$$\psi(\xi) = \begin{cases} \lim_n \psi_n(\xi) & \xi \in C \\ 0 & \xi \notin C \end{cases} \quad \dots \textcircled{10}$$

For each $x \in E$.

$$\psi(f(x)) = \lim_n \underbrace{\psi_n(f(x))}_{z_n(x)} = z(x)$$

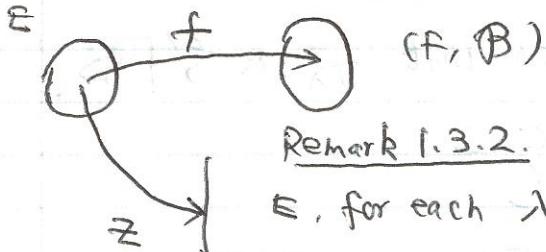
by ⑧ ⑩

i.e. $z(x) = \psi(f(x))$ for all $x \in E$ --- (11)

From (9) (10), we easily see that ψ is \mathcal{B} -meas.

(iii) general case where z is off \mathcal{L} -meas. and \mathbb{R} -valued

Just decompose into positive and negative parts \square



Remark 1.3.2. Given a set E , and a σ -alg \mathcal{G}_λ over E , for each λ in some "Index set" Λ

$$\text{Put } C = \{A \in E : A \in \mathcal{G}_\lambda \text{ for some } \lambda \in \Lambda\}$$

$$= \bigcup_{\lambda \in \Lambda} \mathcal{G}_\lambda$$

Generally, C is NOT a σ -alg. over E .

Observe $\mathcal{G}_\lambda \subset C$, true for each $\lambda \in \Lambda$

$$\text{put } \mathcal{S} = \sigma\{C\}$$

Then (i) \mathcal{S} is a σ -alg.

(ii) $\mathcal{G}_\lambda \subset \mathcal{S}$ for each $\lambda \in \Lambda$

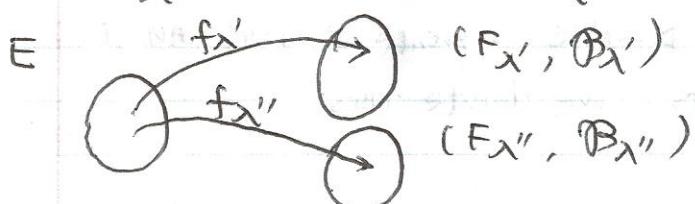
(iii) If \mathcal{S}' is any σ -alg over E st. $\mathcal{G}_\lambda \subset \mathcal{S}'$ for every $\lambda \in \Lambda$, then $\mathcal{S} \subset \mathcal{S}'$

Write $\{\mathcal{G}_\lambda, \lambda \in \Lambda\}$ to denote the σ -alg \mathcal{S}

Next, suppose: E is a set $(F_\lambda, \mathcal{B}_\lambda)$ is σ meas. space

for each λ in an index set Λ

$$f_\lambda : E \rightarrow (F_\lambda, \mathcal{B}_\lambda) \text{ for each } \lambda \in \Lambda$$



We must construct a σ -algebra \mathcal{S} over E st.

common for each λ

$f f_\lambda$ is $\mathcal{S} | \mathcal{B}_\lambda$ for each $\lambda \in \Lambda$

put $\mathcal{G}_\lambda = \sigma\{f_\lambda\} = \{f_\lambda^{-1}(B) : B \in \mathcal{B}_\lambda\}$

$$\boxed{\mathcal{S} = \sigma\{\mathcal{G}_\lambda, \lambda \in \Lambda\}}$$

(i) \mathcal{S} is a σ -alg.

(ii) $\mathcal{S} \cdot f_\lambda$ is $\mathcal{S} |$ meas. for each $\lambda \in \Lambda$

(iii) If \mathcal{S}' is any σ -alg over E such that f_λ is $\mathcal{S}' | \mathcal{B}_\lambda$ -

meas for each $\lambda \in \Lambda$, then $\mathcal{S} \subseteq \mathcal{S}'$.

Write $\sigma\{f_\lambda, \lambda \in \Lambda\}$ to denote \mathcal{S}

$\underbrace{\sigma\{f_\lambda, \lambda \in \Lambda\}}$ — common notation, not quite
 $\underbrace{\sigma\{f_\lambda, \lambda \in \Lambda\}}$ consistent with

Dob. Kolmogorov ←
Levin Ito

Bypass prop. 1.3.3. to remark 1.3.6

§ 1.3.2 σ -algebras on a function space.

Defn. 1.3.7 Given sets F and Λ we write

F^Λ to denote the collection of all mappings from Λ into F .

Example: $F = \mathbb{R}$, $\Lambda = \{1, 2, \dots, n\}$

then F^Λ is the set of all mappings from Λ to F ,

each such mapping corresponds to a member of \mathbb{R}^n .

i.e. $F^\Lambda = \mathbb{R}^n$

Example 2: $F = \mathbb{R}$, $\Lambda = \{0, 1, 2, \dots\}$

each member of F^Λ is a mapping from $\{0, 1, 2, \dots\}$
into \mathbb{R} .

Hence is a sequence $\{x_0, x_1, x_2, \dots\}$ of real numbers.

i.e. F^Λ is the collection of all possible sequence of real numbers
indexed by the non-negative integers. We denote this

By \mathbb{R}^∞ i.e. $F^\Lambda = \mathbb{R}^\infty$

Example (iii) $F = \mathbb{R}$, $\mathcal{L} = [0, \infty)$

i.e. $F^{\mathcal{L}}$ is the set of all \mathbb{R} -valued mappings on the non-negative real line.

Given a set \mathcal{L} and a measurable space (F, \mathcal{B}) .

our goal is to use the σ -alg \mathcal{B} on F to somehow define a useful σ -alg. on the function space $F^{\mathcal{L}}$

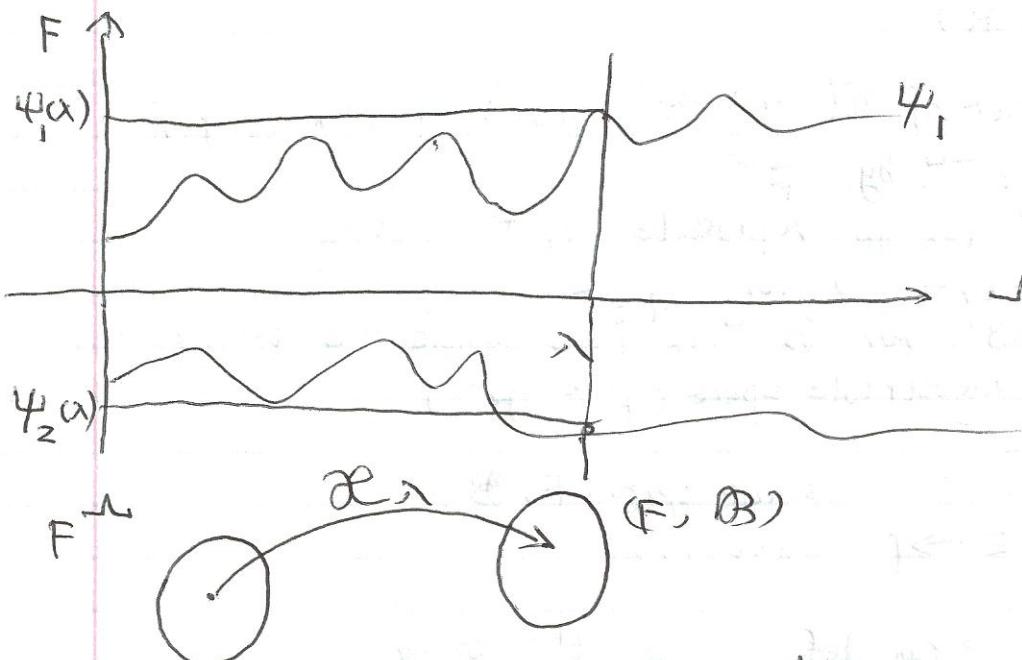
Definition 1.3.9. Let F and \mathcal{L} be given sets.

For each $\lambda \in \mathcal{L}$ define a mapping

$\mathcal{X}_{\lambda}: F^{\mathcal{L}} \rightarrow F$ as follows.

$\mathcal{X}_{\lambda}(\psi) \stackrel{\text{def}}{=} \psi(\lambda), \text{ for all } \psi \in F^{\mathcal{L}}$

e.g. $F = \mathbb{R}$, $\mathcal{L} = [0, \infty)$



Define $\sigma\{\mathcal{X}_{\lambda}, \lambda \in \mathcal{L}\}$ on $F^{\mathcal{L}}$

We take this to be the "natural" σ -alg. on $F^{\mathcal{L}}$ which we denote by $\mathcal{B}^{\mathcal{L}}$ i.e.

$\mathcal{B}^{\mathcal{L}} \stackrel{\text{def}}{=} \sigma\{\mathcal{X}_{\lambda}, \lambda \in \mathcal{L}\}$ since we have used the set

Λ and meas. space (F, \mathcal{B}) to construct the meas. space $(F^\Lambda, \mathcal{B}^\Lambda)$

Jan 12th, 2009 STAT 902.

Given a set Λ and a meas. space (F, \mathcal{B})

F^Λ

Define for each $\lambda \in \Lambda$,

$\mathcal{E}_\lambda: F^\Lambda \rightarrow F$ a mapping such that $\mathcal{E}_\lambda(\bar{x}) = x_\lambda$

as follows $\mathcal{E}_\lambda(\bar{x}) \triangleq x_\lambda$ evaluation mapping

$F^\Lambda \xrightarrow{\mathcal{E}_\lambda} F$ Define: a σ -alg on F^Λ by

$$\mathcal{B}^\Lambda \triangleq \sigma\{\mathcal{E}_\lambda, \lambda \in \Lambda\}$$

(Kolmogorov 1927)

Remark 1.3.17. $\Lambda \triangleq \{0, 1, 2, \dots\}$

and given (F, \mathcal{B})

Then F^Λ is the set of all infinite sequences of elements of F . We normally denote F^Λ by F^∞ .

i.e., each $x \in F^\infty$ can be represented by the sequence

$$x = (x_0, x_1, x_2, \dots) \text{ for } x_i \in F$$

Likewise, write \mathcal{B}^∞ for \mathcal{B}^Λ on F^∞ where $\Lambda = \{0, 1, 2, \dots\}$

i.e. we have a new measurable space $(F^\infty, \mathcal{B}^\infty)$

Remark 1.3.18.

given a set E a measurable space (F, \mathcal{B})

mappings $f_n: E \rightarrow F \quad n=0, 1, 2, \dots$

$E \xrightarrow{f_n} (F, \mathcal{B})$

We can define on E the σ -alg. of $f_n, n=0, 1, 2, \dots$

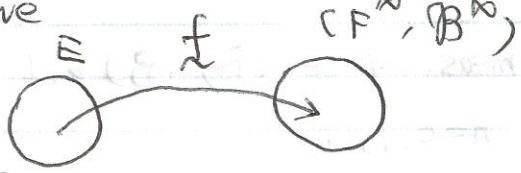
We can also define a mapping

$f: E \rightarrow F^\infty$ as follows:

$$f(x) \triangleq (f_0(x), f_1(x), f_2(x), \dots)$$

$$x \in E$$

i.e. We have



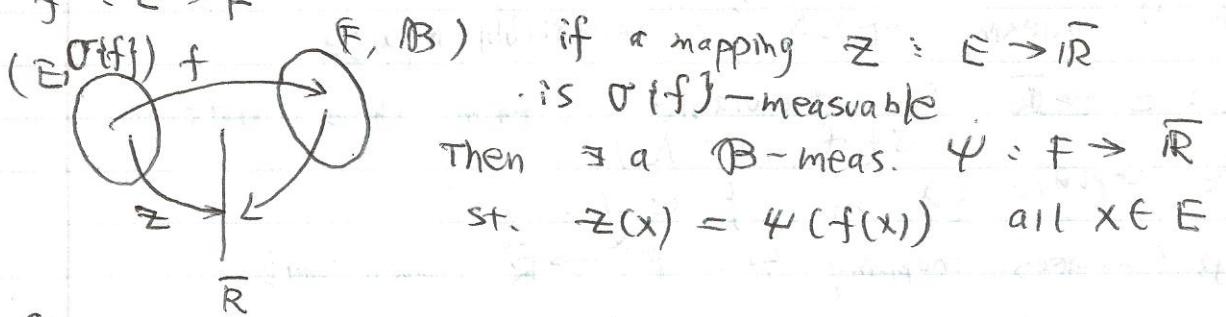
we can define σ -alg on E

$$\sigma\{\tilde{f}\} = \{\tilde{f}^{-1}(B) : B \in \mathcal{B}^\infty\}$$

Prop. 1.3.19. $\sigma\{f_n, n=0, 1, \dots\} = \sigma\{\tilde{f}\}$

Recall the Doob thm 1.3.1

Given a set E , a measurable space (F, \mathcal{B}) and a mapping $f: E \rightarrow F$



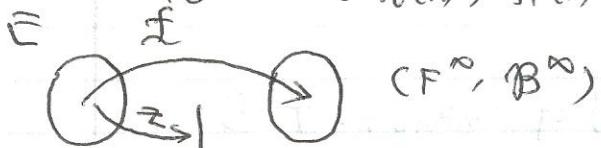
if a mapping $z: E \rightarrow \mathbb{R}$
is $\sigma\{f\}$ -measurable

Then \exists a \mathcal{B} -meas. $\psi: F \rightarrow \mathbb{R}$
st. $z(x) = \psi(f(x))$ all $x \in E$.

Given a set E , a measurable space (F, \mathcal{B}) and a sequence of mapping $f_n: E \rightarrow (F, \mathcal{B})$ $n=0, 1, 2, \dots$

Define a mapping $\tilde{f}: E \rightarrow F^\infty$ as follows:

$$\tilde{f}(x) \stackrel{\text{def}}{=} (f_0(x), f_1(x), f_2(x), \dots) \quad x \in E$$

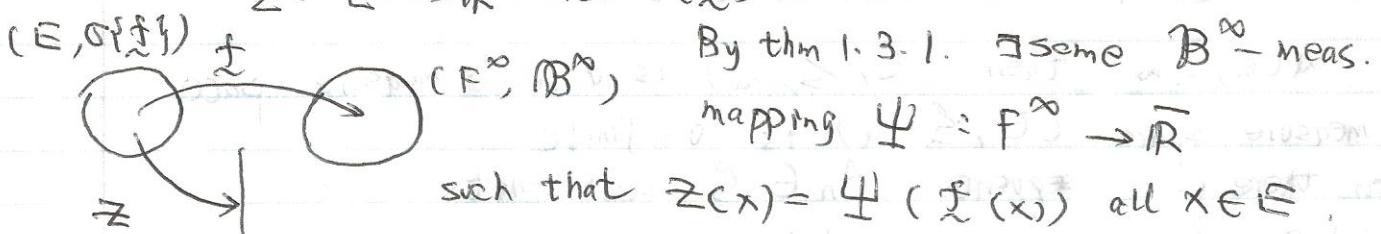


suppose that

$$z: E \rightarrow \mathbb{R} \text{ is } \sigma\{f_n, n=0, 1, 2, \dots\}$$

i.e. by prop. 1.3.19. have that

$$z: E \rightarrow \mathbb{R} \text{ is } \sigma\{\tilde{f}\}-\text{meas.}$$



By thm 1.3.1. \exists some \mathcal{B}^∞ -meas.

mapping $\psi: F^\infty \rightarrow \mathbb{R}$

such that $z(x) = \psi(\tilde{f}(x))$ all $x \in E$,

$$\text{i.e. } z(x) = \psi(f_0(x), f_1(x), f_2(x), \dots) \text{ all } x \in E$$

Thm 1.3.20. Given a set E , a meas. space (F, \mathcal{B}) and mappings $f_n : E \rightarrow (F, \mathcal{B})$ $n=0, 1, 2, \dots$

If $Z : E \rightarrow \overline{\mathbb{R}}$ is $\sigma\{f_n, n=0, 1, \dots\}$ -measurable,

then $\exists \mathcal{B}^\infty$ -meas. mapping ψ from $F^\infty \rightarrow \overline{\mathbb{R}}$ st.

$$Z(x) = \psi(f_0(x), f_1(x), \dots), \quad x \in E$$

Thm 1.3.24. Given a set E , a measurable space (F, \mathcal{B}) and mappings $f_\lambda : E \rightarrow (F, \mathcal{B})$ $\lambda \in \Lambda$

in which the indexing set " Λ " is uncountably infinite

If $Z : E \rightarrow \overline{\mathbb{R}}$ is -measurable, then there exists an infinite sequence $\{\lambda_0, \lambda_1, \dots\} \subset \Lambda$

and a \mathcal{B}^∞ -meas. mapping $\psi : F^\infty \rightarrow \overline{\mathbb{R}}$ such that

$$Z(x) = \psi(f_{\lambda_0}(x), f_{\lambda_1}(x), f_{\lambda_2}(x), \dots) \text{ for all } x \in E$$

§ 1.2.2 σ-alg

Let (E, \mathcal{S}) be a measurable space.

A measure on \mathcal{S} is a mapping $\mu : \mathcal{S} \rightarrow [0, \infty]$

with the following properties.

$$(i) \mu(\emptyset) = 0$$

(ii) For any sequence $\{A_n, n=0, 1, 2, \dots\}$ of members of \mathcal{S} st. $A_m \cap A_n = \emptyset$ where $m \neq n$, we have

$$\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mu(A_n)$$

We call the triple (E, \mathcal{S}, μ) a measure space.

When $\mu(E) = 1$ then (E, \mathcal{S}, μ) is a prob. space.

$\mu(E) < \infty$ then (E, \mathcal{S}, μ) is a finite measure space

A measure space (E, \mathcal{S}, μ) is σ-finite

when there is a sequence $A_n \in \mathcal{S}$ such that

$$(i) E = \bigcup_n A_n \quad (ii) \mu(A_n) < \infty$$

Let (E, \mathcal{S}, μ) be a measure space.

We must define a notion of "integral" for a \mathcal{S} -meas. function $f: E \rightarrow \overline{\mathbb{R}}$.

(i) take $f(x) = I_A(x) \quad x \in E$, for some $A \in \mathcal{S}$.

Define $\int_E f \cdot d\mu \triangleq \mu(A)$

(ii) take $f: E \rightarrow \overline{\mathbb{R}}$ to be a simple \mathcal{S} -meas. function

$$\text{re. } f(x) = \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i}(x) \quad x \in E$$

where $\alpha_i \in \mathbb{R}$, $A_i \in \mathcal{S}$, $i=1, \dots, n$

Define $\int_E f \cdot d\mu = \sum_{i=1}^n \alpha_i \mu(A_i)$

(iii) take $f: E \rightarrow [0, \infty]$ to be \mathcal{S} -measurable.

by prop. 1.2.9 \exists a sequence $f_n: E \rightarrow [0, \infty)$ of \mathcal{S} -meas.

simple functions. st. $0 \leq f_n(x) \leq f_{n+1}(x)$ all $x \in E$

$$f_n(x) \uparrow f(x) \quad \forall x \in E.$$

we define $\int_E f \cdot d\mu \triangleq \lim_{n \rightarrow \infty} \int_E f_n \cdot d\mu$

Alternatively, we can define $\int_E f \cdot d\mu \triangleq \sup \{ \int_E g \cdot d\mu \}$ already defined by part (ii)

$$\int_E f \cdot d\mu \triangleq \sup_g \{ \int_E g \cdot d\mu \}$$

where the supremum is taken over all \mathcal{S} -meas simple functions

$g: E \rightarrow [0, \infty)$ st $0 \leq g(x) \leq f(x)$ all $x \in E$

Emphasis: The integral $\int_E f \cdot d\mu$ of $[0, \infty]$ -valued \mathcal{S} -meas.

function is always defined and has values in $[0, \infty]$

(IV) Take $f: E \rightarrow \overline{\mathbb{R}}$. positive part of f negative part of f

$$\text{put } f_+(x) = \max \{ f(x), 0 \} \quad f_-(x) = \max \{ -f(x), 0 \}$$

Then

$$f(x) = f_+(x) - f_-(x) \quad x \in E$$

$$(f)(x) = f_+(x) + f_-(x) \quad x \in E.$$

Moreover, f_+ , f_- and $|f|$ are $[0, \infty]$ -valued and \mathcal{S} -meas.

Thus the integral $\int_E f(x) d\mu$, $\int_E f_+ d\mu$, $\int_E f_- d\mu$

are defined with values in $[0, \infty]$
we define

$$\int_E f d\mu \triangleq \int_E f_+ d\mu - \int_E f_- d\mu \text{ provided that}$$

$$\int_E f_+ d\mu < \infty \text{ and } \int_E f_- d\mu < \infty \quad (*)$$

In which case, $\int_E f d\mu \in \mathbb{R}$.

The integral $\int_E f d\mu$ is undefined if either of the conditions

(*) fails.

Note (*) is equivalent to $\int_E |f| d\mu < \infty$.

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Section 1.4. probability

① prob. space is a meas. space (Ω, \mathcal{F}, P) st. note-real only
 $P(\Omega) = 1$

a Random variable is a \mathcal{F} -meas. function $X: (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

"random vector" $\cdots \mathcal{F}/\mathcal{B}(\mathbb{R}^d)$ -meas $\cdots X: (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$
can always be expanded uniquely as

$X = (X_1, X_2, \dots, X_d)$ for random variables X_i

Given a random var $X: (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ define the

cdf $F(x) \triangleq P(\{w \in \Omega : X(w) \leq x\}), x \in \mathbb{R}$.

Given a random vector $X: (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}^d$, define the

cdf $F(x^1, x^2, \dots, x^d) \triangleq P(\bigcap_{i=1}^d \{w : X_i(w) \leq x^i\})$

Given a non-negative random variable $X: (\Omega, \mathcal{F}, P) \rightarrow [0, \infty)$

the expectation is always defined and given by

$$EX \triangleq \int_{\Omega} X \cdot dP$$

We have $EX \in [0, \infty]$

Given a rv. $X: (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$

the expected value $E[X]$ is defined only when

$$E|X| < \infty \text{ in which case:} \sum_n |a_n| = \infty \text{ prevent non-absolute convergence}$$

$$EX = \int_{\Omega} X \cdot dP$$

i.e. EX is \mathbb{R} -valued whenever it exists.

for a rv. X suppose either

$$(i) X \geq 0 \text{ or } (ii) E|X| < \infty$$

We then define

$$E[X; A] = \int_{\Omega} X I_A \cdot dP \quad \forall A \in \mathcal{F}$$

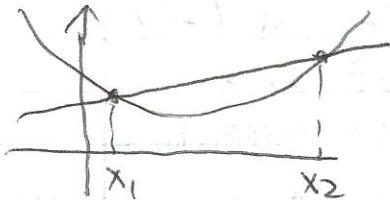
Typon's inequality

i.e. "X is integrable"

suppose X is a rv. st. $E|X| < \infty$ and $c: \mathbb{R} \rightarrow \mathbb{R}$ is a convex function.

i.e. for $x_1, x_2 \in \mathbb{R}$, $\alpha \in [0, 1]$

$$c(\alpha x_1 + (1-\alpha)x_2) \leq \alpha c(x_1) + (1-\alpha)c(x_2)$$



$$\text{st. } E[c(X)] < \infty$$

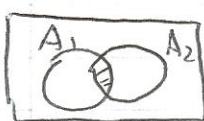
$$c(x) = x^2$$

$$c(X) \leq E[c(X)]$$

§ 1.4.2. Independence

(i) Events A_1 & A_2 in a prob. space (Ω, \mathcal{F}, P) (i.e. $A_i \in \mathcal{F}$) are P -independent.

$$P[A_1 A_2] = P(A_1) \cdot P(A_2)$$



(ii) events $\{A_1, A_2, \dots, A_n\}$ in (Ω, \mathcal{F}, P) (i.e. $A_i \in \mathcal{F}$) are P -independent when for each possible sub-collection $\{A_{i_1}, A_{i_2}, \dots, A_{i_m}\} \subset \{A_1, \dots, A_n\}$

we have

$$P[\bigcap_{k=1}^m P(A_{i_k})] = \prod_{k=1}^m P(A_{i_k}) \quad (\text{17th contrary})$$

(iii) Given σ -algebras $\mathcal{G}_1, \dots, \mathcal{G}_m$, for $\mathcal{G}_i \in \mathcal{F}$ in the prob. space (Ω, \mathcal{F}, P) the collection

(20th contrary)
[kolmogorov]

$\{G_1, G_2, \dots, G_m\}$ is P -independent.

when for each and every choice $A_i \in G_i$, $i=1 \dots m$
the resulting collection of events

$\{A_1, \dots, A_m\}$ is P -independent in the sense of (ii)

(IV) Given σ -algebras G_λ , for each λ in an indexing set Λ
st. $G_\lambda \subset \mathcal{F}$, $\lambda \in \Lambda$, the collection

$\{G_\lambda : \lambda \in \Lambda\}$ is P -independent when for each ^{every} finite

subset $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \Lambda$, the resulting collection

$\{G_{\lambda_1}, G_{\lambda_2}, \dots, G_{\lambda_n}\}$ is P -indep. in the sense of (iii)

(V) Given rv's $X_\lambda : (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ for each λ in an indexing set Λ
the set of rv's $\{X_\lambda : \lambda \in \Lambda\}$ is P -independent when

the collection of σ -algebras

$\{\sigma(X_\lambda), \lambda \in \Lambda\}$ is P -independent in the sense of (IV)

Observe that testing for P -indep. of the collection of rv's

$\{\sigma(X_\lambda) : \lambda \in \Lambda\}$ really reduces to testing for P -independence
of each and every finite sub-family $\{X_{\lambda_1}, X_{\lambda_2}, \dots, X_{\lambda_n}\}$

Thm. 1.4.6. (Kac) A set $\{X_1, \dots, X_n\}$ of rv's on (Ω, \mathcal{F}, P)
is P -independent iff.

$$F_X(x_1, \dots, x_n) = \prod_{i=1}^n F_{X_i}(x_i) \quad \text{all } x_i \in \mathbb{R}$$

\Rightarrow immediate.

\Leftarrow Dynkin π - λ Theorem nontrivial

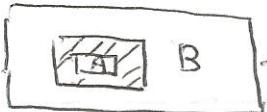
§ 1.5 Monotone Class Theorems:

Recall: a designated collection of subsets $\mathcal{E} \subset \mathcal{P}(\mathbb{E})$ is a π -class
whenever $A + B \in \mathcal{E} \Rightarrow A \cap B \in \mathcal{E}$.

λ -class

Defn. 1.5.2. A designated collection of subsets $\mathcal{E} \subset \mathcal{P}(\mathbb{E})$ is a
 λ -class when

"Dynkin system"



17.

(1) $E \in \mathcal{C}$

(2) when $A+B \in \mathcal{C}$ st. $A \subset B \Rightarrow B \setminus A \in \mathcal{C}$

(3) when $A_n \in \mathcal{C}$, $n=1, 2, \dots$ st. $A_n \subset A_{n+1}$
then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$.

Theorem 1.5.4: Let \mathcal{D} be a π -class of subsets of E

Let \mathcal{C} be a λ -class of subsets of E

st. $\mathcal{D} \subset \mathcal{C}$

then $\sigma(\mathcal{D}) \subset \mathcal{C}$

"Dynkin thm" 1962.

Lemma 1.5.5: Suppose that (Ω, \mathcal{F}, P) is a prob. space and $\mathcal{C} \subset \mathcal{F}$, $\mathcal{D} \subset \mathcal{F}$ st.

(i) \mathcal{D} is a π -class

(ii) \mathcal{C} and \mathcal{D} are P -independent i.e. st. $A \in \mathcal{D}$

then $\sigma(\mathcal{D})$ and \mathcal{C} are P -independent. $B \in \mathcal{C}$

proof: Fix some $B \in \mathcal{C}$. Define $P(AB) = P(A)P(B)$

$$\mathcal{D}_B^* = \{A \in \sigma(\mathcal{D}): P(AB) = P(A)P(B)\}$$

Since \mathcal{D} and \mathcal{C} are P -independent we have

$$P(AB) = P(A)P(B) \quad \text{for all } A \in \mathcal{D}$$

$$\text{i.e. } \mathcal{D} \subset \mathcal{D}_B^* \quad \dots \text{--- ①}$$

Moreover, \mathcal{D}_B^* is a λ -class of events (easy exercise)

By theorem 1.5.4, we get

$$\sigma(\mathcal{D}) \subset \mathcal{D}_B^* \subset \sigma\{\mathcal{D}\}$$

from definition of \mathcal{D}_B^*
From Dynkin's theorem.

$$\text{i.e. } \sigma(\mathcal{D}) = \mathcal{D}_B^*$$

i.e. for each $A \in \sigma\{\mathcal{D}\}$ have $A \in \mathcal{D}_B^*$. hence

$$P(AB) = P(A)P(B)$$

i.e. we have $P(AB) = P(A)P(B)$ for each $B \in \mathcal{C}$.

$$A \in \sigma\{\mathcal{D}\}$$

i.e. $\sigma\{\mathcal{D}\}$ and \mathcal{C} are P -independent \blacksquare

Return to theorem 1.4.6: (mini-version)

If X and Y are random variables on (Ω, \mathcal{F}, P)

$$F_{(X,Y)}(x,y) = F_X(x) F_Y(y) \quad \text{all } x, y \in \mathbb{R}$$

then $\sigma\{X\}$ and $\sigma\{Y\}$ are P -independent

i.e. $\{X, Y\}$ is P -independent.

Lemma 1.5.6. Let \mathcal{D}_1 and \mathcal{D}_2 be P -indep. π -classes of events in (Ω, \mathcal{F}, P) , i.e.

$$\mathcal{D}_1 \subset \mathcal{F}, \quad \mathcal{D}_2 \subset \mathcal{F}$$

$$\& A_1 \in \mathcal{D}_1, \quad A_2 \in \mathcal{D}_2 \Rightarrow P(A_1 A_2) = P(A_1) P(A_2)$$

then $\sigma\{\mathcal{D}_1\}$ and $\sigma\{\mathcal{D}_2\}$ are P -independent.

Proof: From Lemma 1.5.5. we have

$\sigma\{\mathcal{D}_1\}$ and $\sigma\{\mathcal{D}_2\}$ are P -independent.

By Lemma 1.5.5. we have that

$\sigma\{\mathcal{D}_2\}$ and $\sigma\{\mathcal{D}_1\}$ are P -independent. \square

Proof: Put $\mathcal{D}_1 = \{ \{x \leq a\} : a \in \mathbb{R}\}$

$$\{w \in \Omega : x(w) \leq a\}$$

$$\mathcal{D}_2 = \{ \{y \leq b\} : b \in \mathbb{R}\}$$

$$\text{Observe } \{x \leq a\} \cap \{x \leq a_2\} = \{x \leq a, \& a_2\}$$

i.e. \mathcal{D}_1 is a π -class.

Similarly, \mathcal{D}_2 is a π -class.

Moreover, $\sigma\{X\} = \sigma\{\mathcal{D}_1\}$

$$\sigma(Y) = \sigma(\mathcal{D}_2)$$

so we must show that the π -classes \mathcal{D}_1 and \mathcal{D}_2 are indep.

$$P(\{x \leq a\} \cap \{y \leq b\}) = F_{(X,Y)}(a, b) = F_X(a) \cdot F_Y(b)$$

$$= P(\{x \leq a\}) \cdot P(\{y \leq b\})$$

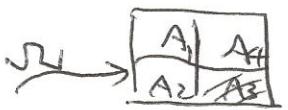
i.e. \mathcal{D}_1 and \mathcal{D}_2 are independent.

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§ 1.4.5 Conditional expectation

Gives (1) a rv. $X = (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$. $E[X] < \infty$

(2) a finite partition $\mathcal{C} = \{A_1, A_2, \dots, A_n\}$ of events $A_i \in \mathcal{F}$.



st. $P(A_t) > 0$ i.e. $A_t \cap A_j = \emptyset$ when $i \neq j$

$$\mathcal{G} = \bigcup_{i=1}^n A_i$$

Define $Z: \Omega \rightarrow \mathbb{R}$ as follows:

for each A_i

$$Z(w) \stackrel{\Delta}{=} \frac{1}{P(A_i)} \int_{A_i} X \cdot dP = \frac{E[X; A_i]}{P(A_i)} \text{ for all } w \in A_i,$$

$$\text{Put } \mathcal{G} = \sigma(\mathcal{C})$$

from measure theory

claim (1) Z is constant over each A_i , hence Z is \mathcal{G} -meas.

$$(2) E[Z; A_i] = \int_{\Omega} Z \cdot 1_{A_i} \cdot dP = \int_{A_i} \frac{1}{P(A_i)} E[X; A_i] \cdot dP \\ = \frac{1}{P(A_i)} E[X; A_i] P(A_i) = E[X; A_i], i=1, \dots, n$$

$$E[Z; A] = E[X; A] \text{ for each } A \in \mathcal{G}$$

In short, we have constructed a \mathcal{G} -meas. func. $Z: \Omega \rightarrow \mathbb{R}$

$$\text{st. } E[Z; A] = E[X; A] \quad A \in \mathcal{G}$$

We call Z the conditional expectation of X given the σ -algebra \mathcal{G} generated by the finite partition $\mathcal{C} = \{A_1, \dots, A_n\}$

$$Z(w) \stackrel{\Delta}{=} \frac{1}{P(A_i)} \int_{A_i} X \cdot dP \quad w \in A_i, i=1, \dots, n$$

Theorem 1.4.13. Suppose $X: (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ st. $E|X| < \infty$ and suppose

that $\mathcal{G} \subset \mathcal{F}$ is a σ -algebra. Then \exists a \mathcal{G} -measurable rv. Z

: $\Omega \rightarrow \mathbb{R}$ st. $E|Z| < \infty$ and

$$E[Z; A] = E[X; A], \quad A \in \mathcal{G}$$

furthermore, if $\tilde{Z}: \Omega \rightarrow \mathbb{R}$ is any \mathcal{G} -measurable rv. i.e. $E|\tilde{Z}| < \infty$

$$\text{and } E[\tilde{Z}; A] = E[X; A] \quad A \in \mathcal{G}$$

$$\text{then } P\{w \in \Omega : Z(w) \neq \tilde{Z}(w)\} = 0$$

We use $E[X|\mathcal{G}]$ to denote a fixed but arbitrary choice of a func. Z given by thm. 1.4.13.

Thm 1.4.15. Given rvs $X, Y : (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$, st. $E|X| < \infty$

and a sub- σ -algebra $\mathcal{G} \subset \mathcal{F}$. Then

(a) If $X \geq 0$ a.s. $E[X|\mathcal{G}] \geq 0$ a.s.

(b) If $\alpha, \beta \in \mathbb{R}$, $E|Y| < \infty$, then

$$E[\alpha X + \beta Y | \mathcal{G}] = \alpha E[X|\mathcal{G}] + \beta E[Y|\mathcal{G}] \text{ a.s.}$$

(c) If $E|XY| < \infty$, then and Y is \mathcal{G} -meas. then

$$E[XY|\mathcal{G}] = Y \cdot E[X|\mathcal{G}] \text{ a.s.}$$

(d) if $\mathcal{H} \subset \mathcal{G}$ is a sub- σ -algebra of \mathcal{G} , then

$$E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}] \text{ a.s. "tower property"}$$

$$(e) E[E[X|\mathcal{G}]] = EX \text{ two numbers.}$$

(f) if $\mathcal{H} \subset \mathcal{F}$ is a sub- σ -algebra of \mathcal{F} , and $\sigma\{X\}$ and \mathcal{H} are independent, then

$$E[X|\mathcal{H}] = E[X] \text{ a.s.}$$

More generally, when $\mathcal{H} \subset \mathcal{G}$ is a σ -algebra

and $\sigma\{\sigma\{X\}, \mathcal{G}\}$ and \mathcal{H} are independent, then

$$E[X|\sigma\{\mathcal{G}, \mathcal{H}\}] = E[X|\mathcal{G}] \text{ a.s.}$$

Proofs of (a) - (e) are elementary.

Recall the Jensen Inequality Thm 1.4.2.

Let $X : (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ be integrable ($E|X| < \infty$) and let

$c : \mathbb{R} \rightarrow \mathbb{R}$ be convex such that $c(X)$ is integrable. $E[|c(x)|] < \infty$

then $c(EX) \leq E[c(X)]$

Thm 1.4.20. Let $X : (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ be integrable, let $c : \mathbb{R} \rightarrow \mathbb{R}$ be convex st. $c(X)$ is integrable, and let $\mathcal{G} \subset \mathcal{F}$ be a sub- σ -algebra of \mathcal{F}

$$c(E[X|\mathcal{G}]) \leq E[c(X)|\mathcal{G}] \text{ a.s.}$$

i.e. $P\{w \in \Omega : c(E[X|\mathcal{G}](w)) > E[c(X)|\mathcal{G}](w)\} = 0$

From the Lebesgue Dominance Convergence Theorem we have the following

if $X_n, X : (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ are rvs. for all $n = 1, 2, \dots$

st. (i) $\lim X_n = X$ a.s.

i.e. $P\{w : \lim_n X_n(w) = X(w)\} = 1$

(ii) \exists some integrable rv. Z . st. $|X_n| \leq Z$ a.s. for each $n = 1, 2, \dots$

the same Z works for every x_n .

21.

i.e. $P\{w \in \mathbb{R} : |x_n(w)| \leq z(w)\} = 1$

then $\lim_n \mathbb{E}[x_n] = \mathbb{E}[x]$

Thm. 1.4-19 (c) if $x_n, x : (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ are rvs for all $n=1, 2, \dots$ st.

(i) $\lim_n x_n = x$ as

(ii) \exists some integrable y s.t. $|x_n| \leq y$ a.s for each $n=1, 2, \dots$ and $\mathcal{G} \subset \mathcal{F}$ is a sub-sigma-algebra. then

$$\lim_n \mathbb{E}[x_n | \mathcal{G}] = \mathbb{E}[x | \mathcal{G}] \text{ a.s.}$$

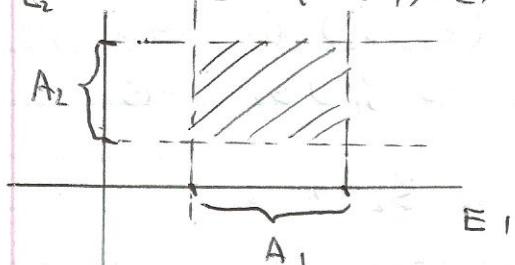
§ 1.2.7. The Fubini-Tonelli Theorem

Let $(E_1, \mathcal{S}_1, \mu_1)$ and $(E_2, \mathcal{S}_2, \mu_2)$ be measure spaces.

Put $E_1 \otimes E_2 \triangleq \{(x_1, x_2) : x_1 \in E_1, x_2 \in E_2\}$

for $A_1 \in \mathcal{S}_1, A_2 \in \mathcal{S}_2$

$$E_1 \otimes A_2 \triangleq \{(x_1, x_2) : x_1 \in A_1, x_2 \in A_2\} \text{ a rectangle.}$$



when $A_1 \in \mathcal{S}_1, A_2 \in \mathcal{S}_2$ then

$A_1 \otimes A_2$ is called a measurable rectangle

$$\text{put } \mathcal{C} \triangleq \{A_1 \otimes A_2 : A_1 \in \mathcal{S}_1, A_2 \in \mathcal{S}_2\}$$

then we have a σ -alg. $\sigma(\mathcal{C})$

we denote $\sigma(\mathcal{C})$ by $\mathcal{S}_1 \otimes \mathcal{S}_2$

this gives us a measurable space

$$(E_1 \otimes E_2, \mathcal{S}_1 \otimes \mathcal{S}_2)$$

* product of σ -algebras \mathcal{S}_1 and \mathcal{S}_2

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Given measure spaces $(E_1, \mathcal{S}_1, \mu_1)$ and $(E_2, \mathcal{S}_2, \mu_2)$

$$E_1 \otimes E_2 \triangleq \{(x_1, x_2) : x_1 \in E_1, x_2 \in E_2\}$$

Take $A_1 \subset E_1, A_2 \subset E_2$ $A_1 \otimes A_2 \triangleq \{(x_1, x_2) : x_1 \in A_1, x_2 \in A_2\}$
rectangle

If $A_1 \in \mathcal{S}_1$, $A_2 \in \mathcal{S}_2$, then $A_1 \otimes A_2$ is a measurable rectangle.

Put $\mathcal{C} = \{A_1 \otimes A_2 : A_1 \in \mathcal{S}_1, A_2 \in \mathcal{S}_2\}$ π -class

Define $\mathcal{S}_1 \otimes \mathcal{S}_2 = \sigma(\mathcal{C})$

product of \mathcal{S}_1 with \mathcal{S}_2

We have a measurable space

$(E_1 \otimes E_2, \mathcal{S}_1 \otimes \mathcal{S}_2)$

Now use the given measures μ_1 and μ_2 to construct a useful measure μ on $\mathcal{S}_1 \otimes \mathcal{S}_2$

Lemma 1.2.39. Suppose that the measure spaces $(E_1, \mathcal{S}_1, \mu_1)$ and

$(E_2, \mathcal{S}_2, \mu_2)$ are σ -finite, then \exists a unique measure μ on $\mathcal{S}_1 \otimes \mathcal{S}_2$

such that $\mu(A_1 \otimes A_2) = \mu_1(A_1) \mu_2(A_2)$ denoted by $\mu_1 \otimes \mu_2$

This we have a measure space

$(E_1 \otimes E_2, \mathcal{S}_1 \otimes \mathcal{S}_2, \mu_1 \otimes \mu_2)$ — also σ -finite

Suppose that (E_1, \mathcal{S}_1) and (E_2, \mathcal{S}_2) are measurable.

Then suppose $f: E_1 \otimes E_2 \rightarrow \bar{\mathbb{R}}$ is $\mathcal{S}_1 \otimes \mathcal{S}_2$ -meas.

Then for each $x_1 \in E_1$, the mapping

$x_2 \mapsto f(x_1, x_2): E_2 \rightarrow \bar{\mathbb{R}}$ is \mathcal{S}_2 -meas.

for each $x_2 \in E_2$ the mapping

$x_1 \mapsto f(x_1, x_2): E_1 \rightarrow \bar{\mathbb{R}}$ is \mathcal{S}_1 -meas.

Suppose that $(E_1, \mathcal{S}_1, \mu_1)$ and $(E_2, \mathcal{S}_2, \mu_2)$ are σ -finite meas. spaces.

Then we have "product" measure space

$(E_1 \otimes E_2, \mathcal{S}_1 \otimes \mathcal{S}_2, \mu_1 \otimes \mu_2)$

Take a function $f: E_1 \otimes E_2 \rightarrow [0, \infty]$ to be $\mathcal{S}_1 \times \mathcal{S}_2$ measurable,

then the integral (Lebesgue)

$$\int_{E_1 \otimes E_2} f d(\mu_1 \otimes \mu_2) \text{ exists in } [0, \infty]$$

Now fix $x_1 \in E_1$, then from Lemma 1.2.38 then function $f(x_1, \cdot)$ is \mathcal{S}_2 -meas. and nonnegative

Thus the \mathbb{L} -integral

$$\int_{E_2} f(x_1, x_2) \downarrow d\mu_2(x_2) \text{ exists in } [0, \infty]$$

i.e. we have a mapping $x_1 \rightarrow \int_{E_2} f(x_1, x_2) d\mu_2(x_2) : E_1 \rightarrow [0, +\infty]$

In the same way, we also have a mapping

$$x_2 \rightarrow \int_{E_1} f(x_1, x_2) d\mu_1(x_1) : E_2 \rightarrow [0, +\infty] \dots \textcircled{2}$$

Thm. 1.2.40 The mapping in $\textcircled{1}$ is \mathcal{S}_1 -meas.

The \dots $\textcircled{2}$ is \mathcal{S}_2 -meas.

Moreover, the integrals

$$\int_{E_1} \left\{ \int_{E_2} f(x_1, x_2) d\mu_2(x_2) \right\} d\mu_1(x_1)$$

$$\text{and } \int_{E_2} \left\{ \int_{E_1} f(x_1, x_2) d\mu_1(x_1) \right\} d\mu_2(x_2)$$

$$\text{and } \int_{E_1 \otimes E_2} f(x_1, x_2) d(\mu_1 \otimes \mu_2)$$

ARE EQUAL. \diamond proof on page 47 in Lecture notes.

Thm. 1.2.41 (Fubini) suppose that $(E_1, \mathcal{S}_1, \mu_1)$ and $(E_2, \mathcal{S}_2, \mu_2)$ are σ -finite measure spaces, and $f: E_1 \otimes E_2 \rightarrow \bar{\mathbb{R}}$ is $\mathcal{S}_1 \otimes \mathcal{S}_2$ -meas.

$$\text{If } \int_{E_1 \otimes E_2} |f| d(\mu_1 \otimes \mu_2) < \infty$$

then all the statements of thm. 1.2.40 hold.

(Tonelli \uparrow)

J.L. Doob www-groups.dcs.st-and.ac.uk/~history/Biographies/Doob.html

Chapter 2: Discrete parameter Martingales.

Defn. 2.1.2. a filtration in a prob. space (Ω, \mathcal{F}, P) is a sequence $\{\mathcal{F}_n, n=0, 1, 2, \dots\}$ of σ -algebras \mathcal{F}_n st.

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$$

If $\{X_n, n=0, 1, 2, \dots\}$ is a given sequence of RV's on (Ω, \mathcal{F}, P)

then the sequence of pairs $\{(x_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ is adapted when each x_n is \mathcal{F}_n -meas.

Defn. 2.1.3. Given a filtration $\{\mathcal{F}_n : n=0, 1, 2, \dots\}$ in $(\Omega, \mathcal{F}, \mathbb{P})$

$$\text{Put } \mathcal{F}_\infty = \sigma\{\mathcal{F}_n, n=0, 1, 2, \dots\}$$

\uparrow smallest σ -alg. on Ω , which is large enough to include every \mathcal{F}_n

$$\mathcal{F}_\infty \subset \mathcal{F}$$

Example: 2.1.4: Let $\{y_n, n=0, 1, 2, \dots\}$ be a given sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$

$$\text{Put } \mathcal{F}_n \triangleq \sigma\{y_0, y_1, \dots, y_n\}$$

Then $\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subset \mathcal{F}$.

Now suppose that $\{x_n, n=0, 1, 2, \dots\}$ is a sequence of r.v.'s s.t.

$\{(x_n, \mathcal{F}_n)\}$ is adapted.

i.e. x_n is \mathcal{F}_n -meas. for each $n=0, 1, 2, \dots$

i.e. x_n is $\sigma\{y_0, \dots, y_n\}$ -meas.

i.e. \exists some $B(\mathbb{R}^{n+1})$ -meas. mapping $\phi_n : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ s.t.

$$x_n(w) = \phi_n(y_0(w), y_1(w), \dots, y_n(w)) \text{ all } w \in \Omega$$

i.e. at any w , the value of $x_n(w)$ is completely determined by the values $y_0(w), y_1(w), \dots, y_n(w)$

Defn. 2.1.5. Given a filtration $\{\mathcal{F}_n, n=0, 1, 2, \dots\}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then a mapping

$T : \Omega \rightarrow \{0, 1, 2, \dots +\infty\}$ is a random time when

$\{w : T(w) \leq n\} \in \mathcal{F}$ for all $n=0, 1, 2, \dots$

The mapping $T : \Omega \rightarrow \{0, 1, 2, \dots +\infty\}$ is a \mathcal{F}_n -stopping time when

$\{T \leq n\} \in \mathcal{F}_n \quad n=0, 1, 2, \dots$

Suppose that $\{y_n\}$ is a given sequence of r.v.s on $(\Omega, \mathcal{F}, \mathbb{P})$

Let us define $\mathcal{F}_n \triangleq \sigma\{y_0, \dots, y_n\} \quad n=0, 1, 2, \dots$

suppose that T is a \mathcal{F}_n -stopping time

i.e. $\{T \leq n\} \in \mathcal{F}_n \triangleq \sigma\{Y_0, \dots, Y_n\}$

put $\mathbb{I}_{\{T \leq n\}}(w) = \begin{cases} 1 & \text{when } T(w) \leq n \\ 0 & \text{otherwise} \end{cases}$
 $\underbrace{\mathbb{I}_{\{T \leq n\}}}_{\mathcal{F}_n\text{-meas.}}$

i.e. $\mathbb{I}_{\{T \leq n\}}$ is \mathcal{F}_n -meas

i.e. by Doob-thm. \exists a meas: $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ st.

$$\boxed{\mathbb{I}_{\{T \leq n\}}(w) = \psi(Y_0(w), Y_1(w), \dots, Y_n(w)) \quad \text{for all } w \in \Omega}$$

i.e. to decide whether or not $T(w) \leq n$ (given w)

we only need know the values $Y_0(w), Y_1(w), \dots, Y_n(w)$

Example 2.1.8:

suppose that $\{(X_n, F_n)\}$ is an adapted sequence in (Ω, \mathcal{F}, P)

Define: For each w .

$$S(w) \triangleq \min \{ k \in \{0, 1, 2, \dots\} : X_k(w) \leq a \} \quad a \in \mathbb{R} \text{ given}$$

Define $\min(\emptyset) = +\infty$

i.e. $S(w) \triangleq \infty$ when $X_k(w) > a$ for all $k = 0, 1, 2, \dots$

We show that S is a \mathcal{F}_n -stopping time!

For a fixed w , have

$$X_0(w) \leq a \quad \text{iff} \quad S(w) = 0$$

$$\text{i.e. } \{S=0\} = \{X_0 \leq a\} \in \mathcal{F}_0$$

More generally for any integer $n = 1, 2, \dots$

$$S(w) = n \Leftrightarrow X_k(w) > a \text{ for all } k = 1, 2, \dots, n-1 \quad \& \quad X_n(w) \leq a$$

$$\text{i.e. } \{S=n\} = [\bigcap_{k=0}^{n-1} \{X_k > a\}] \cap \{X_n \leq a\}$$

$$\underbrace{\bigcap_{k=0}^{n-1} \mathcal{F}_k}_{\mathcal{F}_n} \subset \mathcal{F}_n \quad \& \quad \mathcal{F}_n \text{ is a } \sigma\text{-algebra}$$

$$\therefore \{S=n\} \in \mathcal{F}_n \text{ for all } n = 0, 1, 2, \dots$$

$$\{S \leq m\} = \bigcup_{n=1}^m \{S = n\} \in \mathcal{F}_m$$

what if $\max_{T(w)} \mathcal{F}_n \subset \mathcal{F}_m$

$$T(w) \triangleq \sup \{k \in \{0, 1, 2, \dots\} : X_k(w) \leq a\} \quad a \in \mathbb{R} \text{ given}$$

not a stopping time

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Defn 2.1.10 Given a \mathcal{F}_n -stopping time T , Define

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq n\} \in \mathcal{F}_n, n=0, 1, 2, \dots, +\infty\}$$

\uparrow
pre- σ -algebra of T

Prop. 2.1.11. (a) \mathcal{F}_T is a σ -alg. over \mathcal{F} , $\mathcal{F}_T \subset \mathcal{F}_\infty \subset \mathcal{F}$

(b) T is \mathcal{F}_T -meas. i.e. $\sigma\{T\} \subset \mathcal{F}_T$

(c) $\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T=n\} \in \mathcal{F}_n, n=0, 1, 2, \dots, +\infty\}$

(d) if $T(w) \equiv n$ for all w , then $\mathcal{F}_T = \mathcal{F}_n$

prop. 2.1.12,

Let $S \& T$ be \mathcal{F}_n -stopping times, then

(a) $S \wedge T, S \vee T, S+T$ are \mathcal{F}_n -stopping times.

$$(S \wedge T)(w) \triangleq \min\{S(w), T(w)\}$$

$$(S \vee T)(w) \triangleq \max\{S(w), T(w)\}$$

(b) If $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$

(c) $\mathcal{F}_{S \wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$

(d) $\{S < T\}, \{S = T\}, \{S > T\}$ are members of $\mathcal{F}_{S \wedge T}$

(e) If $A \in \mathcal{F}_S$, then $A \cap \{S \leq T\}, A \cap \{S < T\}, A \cap \{S = T\}$ are members of $\mathcal{F}_{S \wedge T}$

Proof of (d): First show $A \cap \{S \leq T\} \in \mathcal{F}_T$ for $A \in \mathcal{F}_S$

Fix some $A \in \mathcal{F}_S$. Observe

$$\{S \leq T\} \cap \{T \leq n\} \subset \{S \leq n\} \quad \text{--- (1)}$$

$$\{S \leq n\} \cap \{T \leq n\} \cap \{S \leq T\} = \{S \leq n\} \cap \{T \leq n\} \cap \{S \leq n \leq T \leq n\} \quad \text{--- (2)}$$

Look at we must show

$$(A \cap \{S \leq T\}) \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n = 0, 1, 2, \dots, +\infty$$

to conclude that $A \cap \{S \leq T\} \in \mathcal{F}_T$

then

$$\begin{aligned} (A \cap \{S \leq T\}) \cap \{T \leq n\} &= A \cap \{S \leq T\} \cap \{T \leq n\} \cap \{S \leq n\} \quad (1) \\ &= (A \cap \{S \leq n\}) \cap \{T \leq n\} \cap \{S \leq T\} \quad (2) \\ &= A \underbrace{\cap \{S \leq n\}}_{\in \mathcal{F}_n} \cap \underbrace{\{T \leq n\}}_{\in \mathcal{F}_n} \cap \underbrace{\{S \leq T\}}_{\in \mathcal{F}_n} \dots (3) \end{aligned}$$

since $A \in \mathcal{F}_S$ we have $\underbrace{A \cap \{S \leq n\}}_{\in \mathcal{F}_n} \in \mathcal{F}_n \dots (4)$

also. $\{T \leq n\} \in \mathcal{F}_n$ (T is stopping time) ... (5)

Moreover, the stopping time $S \wedge n$ is $\mathcal{F}_{S \wedge n}$ -meas. (by prop. 2.1.11(b))

hence since $\mathcal{F}_{S \wedge n} \subset \mathcal{F}_n$ ie. $S \wedge n$ is \mathcal{F}_n -meas.

similarly, $T \wedge n$ is \mathcal{F}_n -measurable

i.e. $\{S \wedge n \leq T \wedge n\} \in \mathcal{F}_n \dots (6)$

i.e. From (3)-(6) $A \cap \{S \leq T\} \cap \{T \leq n\} \in \mathcal{F}_n \quad n=0, 1, 2, \dots$

$A \cap \{S \leq T\} \in \mathcal{F}_T \dots (7)$ for each $A \in \mathcal{F}_S$

In particular, ($A = \cup$) we have

$$\{S \leq T\} \in \mathcal{F}_T \quad \{S > T\} \in \mathcal{F}_T \dots (8)$$

We next show $\{S > T\} \in \mathcal{F}_S \dots (9)$

put $R \stackrel{A}{=} S \wedge T \dots (10)$

Then R is a stopping time, hence R is \mathcal{F}_R -meas.

But $\mathcal{F}_R \subset \mathcal{F}_S$ since $(R \leq S)$ hence R is \mathcal{F}_S -meas. ... (11)

Moreover, S is \mathcal{F}_S -meas, thus

$$\{R < S\} \in \mathcal{F}_S \dots (12)$$

$$\text{But } \{R < S\} \stackrel{(10)}{=} \{S \wedge T < S\} = \{T < S\} \dots (13)$$

$$\text{From (12) (13)} \quad \{T < S\} \in \mathcal{F}_S \dots (14)$$

$$\text{From (14), (8)} \quad \{T < S\} \in \mathcal{F}_S \cap \mathcal{F}_T = \mathcal{F}_{S \wedge T} \dots (15)$$

By Symmetry,

$$\{S < T\} \in \mathcal{F}_T \cap \mathcal{F}_S = \mathcal{F}_{S \wedge T} \dots (16)$$

$$\{S = T\} = \underbrace{\{S < T\}^c}_{\in \mathcal{F}_{S \wedge T}} \cap \underbrace{\{T < S\}^c}_{\in \mathcal{F}_{S \wedge T}} \in \mathcal{F}_{S \wedge T}$$

Doeblin / Dynkin

§2.2. Martingales, sub-supermartingale

Defn: 2.2.1. A supermartingale (submartingale, martingale) is an adapted sequence $\{X_n, \mathcal{F}_n\}$, $n=0, 1, 2, \dots$

st.

$$(a) \mathbb{E}[|X_n|] < \infty \text{ for all } n.$$

$$(b) \mathbb{E}[X_n | \mathcal{F}_m] \leq X_m \text{ a.s. for all } m \leq n$$

[\leq instead of \leq]

[$=$ instead of \leq]

E.g. 2.2.5. Given a filtration $\{\mathcal{F}_n, n=0, 1, 2, \dots\}$

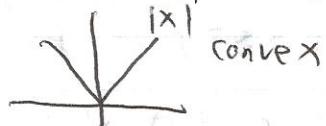
and integrable random variable X on (Ω, \mathcal{F}, P) (i.e. $\mathbb{E}|X| < \infty$)

$$\text{Put } X_n = \mathbb{E}[X | \mathcal{F}_n]$$

clearly, X_n is \mathcal{F}_n -meas. i.e. (X_n, \mathcal{F}_n) is an adapted sequence.

Moreover, $\mathbb{E}[|X_n|] = \mathbb{E}[|\mathbb{E}[X | \mathcal{F}_n]|]$
by Jensen

$$\leq \mathbb{E}[\mathbb{E}[|X| | \mathcal{F}_n]] = \mathbb{E}[|X|] < \infty$$



therefore $\mathbb{E}[|X_n|] < \infty$

(b) Fix $m \leq n$, i.e. $\mathcal{F}_m \subset \mathcal{F}_n$

$$\mathbb{E}[X_n | \mathcal{F}_m] = \mathbb{E}[\mathbb{E}[X | \mathcal{F}_n] | \mathcal{F}_m] = \mathbb{E}[X | \mathcal{F}_m] = X_m$$

i.e. $\mathbb{E}[X_n | \mathcal{F}_m] = X_m$ a.s.

i.e. we have a martingale.

prop. 2.2.10. (a) Given a martingale, $\{(X_n, \mathcal{F}_n)\}$ and a convex function

$c: \mathbb{R} \rightarrow \mathbb{R}$ st $c(X_n)$ is integrable.

Then $\{(c(X_n), \mathcal{F}_n)\}$ is a sub-martingale.

(b) Given a submartingale, $\{(X_n, \mathcal{F}_n)\}$ and a convex function

$c: \mathbb{R} \rightarrow \mathbb{R}$ which is non-decreasing and integrable. Then

$\{(c(X_n), \mathcal{F}_n)\}$ is a sub-martingale.

§2.3. The optional Sampling Theorem (Doob & Halmos)

Let $\{(X_n, \mathcal{F}_n)\}$ be a supermartingale

Then

$$\mathbb{E}[X_n | \mathcal{F}_m] \leq X_m \text{ a.s. when } m \leq n$$

The O.S. Thm says that, when, S, T are stopping times

when $S \leq T$ then

$$\mathbb{E}[X_T | \mathcal{F}_s] \leq X_s \text{ a.s.}$$

Remark 2.3.1. Given a sequence of rvs $\{X_n, n=0, 1, \dots, \infty\}$ and a mapping $T: \Omega \rightarrow \{0, 1, 2, \dots, +\infty\}$

Define $X_T: \Omega \rightarrow \mathbb{R}$ as follows:

$$X_T(\omega) \triangleq X_n(\omega) \text{ for all } \omega \in \{T=n\}$$

Note that $\{T=i\} \cap \{T=m\} = \emptyset$ when $m \neq i$.

$$\Omega = \bigcup_{0 \leq n \leq \infty} \{T=n\}$$

Prop. 2.3.2: Given (i) an adapted sequence

$$\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$$

(ii) the \mathcal{F}_∞ -meas. $X_\infty: \Omega \rightarrow \mathbb{R}$

(iii) $T: \Omega \rightarrow \{0, 1, 2, \dots, \infty\}$ is a \mathcal{F}_n -stopping time.

Then X_T is \mathcal{F}_T -meas.

Proof: We must show

$$\{\omega \in \Omega : X_T(\omega) \in I\} \in \mathcal{F}_T \text{ for all } I \in \mathcal{B}(\mathbb{R})$$

Fix some $I \in \mathcal{B}(\mathbb{R})$. By def. of X_T , we have

$$\{X_T \in I\} \cap \{T=n\} = \underbrace{\{X_n \in I\}}_{\in \mathcal{F}_n} \cap \underbrace{\{T=n\}}_{\in \mathcal{F}_n}$$

for all $n=0, 1, 2, \dots, +\infty$

i.e. $\underbrace{\{X_T \in I\}}_{\in \mathcal{F}_T} \cap \{T=n\} \in \mathcal{F}_n \text{ for } n=0, 1, 2, \dots, +\infty$

By prop. 2.1.11(c) we have that

$$\{X_T \in I\} \in \mathcal{F}_T \quad \square$$

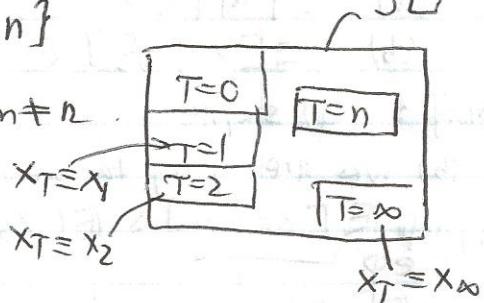
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Remark 2.3.1. Given a sequence of random variables $\{X_n, n=0, 1, 2, \dots, +\infty\}$ and a mapping $T: \Omega \rightarrow \{0, 1, 2, \dots, +\infty\}$

Define $X_T: \Omega \rightarrow \mathbb{R}$ as follows

$$X_T(\omega) \triangleq X_n(\omega) \text{ all } \omega \in \{T=n\}$$

Observe $\{T=n\} \cap \{T=m\} = \emptyset$ $\bigcup_{\{T=n\}} = \Omega$
 $0 \leq n \leq \infty$



Prop. 2.3.2.

Thm 2.3.3. (optional sampling thm - basic version)

Given a supermartingale $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ and \mathcal{F}_n -stopping time $S \leq T$, s.t. $S \leq N$ for some constant $N < \infty$. Then

$$(a) \mathbb{E}|X_T| < \infty$$

$$(b) \mathbb{E}[X_T | \mathcal{F}_S] \leq X_S \text{ a.s.}$$

Proof: (a) simple.

(b) we are going to show that

by defn $\mathbb{E}[X_T; A] \leq \mathbb{E}[X_S; A]$ for each $A \in \mathcal{F}_S$... ①
of C. exp

$$\mathbb{E}[\mathbb{E}[X_T | \mathcal{F}_S]; A] \leq \mathbb{E}[X_S; A]$$

i.e. $\mathbb{E}[\underbrace{\mathbb{E}[X_T | \mathcal{F}_S] - X_S}_{\mathcal{F}_S\text{-meas.}}; A] \leq 0$ for each $A \in \mathcal{F}_S$... ②

[From Thm 1.2.2 (b)] and ② we get

$$\mathbb{E}[X_T | \mathcal{F}_S] - X_S \leq 0 \text{ a.s.}$$

aside $f(\bar{E}, S, u) \rightarrow \bar{R}$
suppose $\int_A f du \leq 0$

Thus we must show ① Fix some $A \in \mathcal{F}_S$.

$$\sum_{n=1}^N \mathbb{I}\{S < n \leq T\}(X_n - X_{n-1})$$

$$= \sum_{n=T+S(w)}^T (X_n - X_{n-1}) = X_T(w) - X_S(w) \quad \dots \text{③}$$

Take expectations over A of each side of (3)

$$\mathbb{E}[X_T - X_S; A] = \mathbb{E}\left\{\sum_{n=1}^N \mathbb{I}\{S < n \leq T\}(X_n - X_{n-1}); A\right\}$$

$$= \sum_{n=1}^N \mathbb{E}[\mathbb{I}\{S < n \leq T\}(X_n - X_{n-1}); A]$$

$$= \sum_{n=1}^N \mathbb{E}[X_n - X_{n-1}; A \cap \{S < n \leq T\}] \quad \dots \text{④}$$

we need to show ≤ 0

$$\text{Now } A \cap \{S < n \leq T\} = A \cap \{S < n\} \cap \{n \leq T\}$$

$$= \underbrace{A \cap \{S \leq n-1\}}_{\in \mathcal{F}_{n-1}} \cap \underbrace{\{n-1 < T\}}_{\mathcal{F}_{n-1}} \quad \dots \text{⑤}$$

Since $A \in \mathcal{F}_s$, have

$$A \cap \{S \leq n-1\} \in \mathcal{F}_{n-1} \quad \text{--- (6)}$$

$$\{n-1 < T\} \subset \{T \leq n\} \in \mathcal{F}_{n-1} \quad \text{--- (7)}$$

$$\therefore A \cap \{S \leq n \leq T\} \in \mathcal{F}_{n-1} \quad \text{--- (8)}$$

since $\{(X_n, \mathcal{F}_n)\}$ is a supermartingale, we know that

$$\mathbb{E}[X_n - X_{n-1} | B] \leq 0 \quad \text{for each } B \in \mathcal{F}_{n-1}$$

(semi-trivial ex)

Take $B = A \cap \{S \leq n \leq T\}$ to get --- (1) □

Defn. 2.3.4. $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ $\mathbb{E}[X_n | \mathcal{F}_m] \leq X_m$

The indexed sequence $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots +\infty\}$ is a closed supermartingale when

(i) $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ is a supermartingale

(ii) $\mathcal{F}_\infty \triangleq \sigma\{\mathcal{F}_n, n=0, 1, 2, \dots\}$

(iii) X_∞ is \mathcal{F}_∞ -meas. such that

$$\mathbb{E}|X_\infty| < \infty \quad \text{and} \quad \mathbb{E}[X_\infty | \mathcal{F}_n] \leq X_n \text{ a.s. for all } n=0, \dots$$

Thm 2.3.9 (O.S Thm - second version)

Suppose that $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots +\infty\}$ is a closed supermartingale and S, T are \mathcal{F}_n -stopping times, st. $S \leq T$. Then

$$(a) \mathbb{E}|X_T| < \infty$$

$$(b) \mathbb{E}[X_T | \mathcal{F}_S] \leq X_S \text{ a.s.}$$

Thm 2.3.14.

Suppose that $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ is a martingale and T is a \mathcal{F}_n -stopping time.

Define $X^T_{n(w)} \triangleq \begin{cases} X_n(w), & n=0, 1, 2, \dots T(w) \\ X_{T(w)}(w), & n > T(w) \end{cases}$

Then $\{(X^T_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ is a martingale.

§2.4 Thm 2.4.4: Suppose that $p \in (1, \infty)$ is a constant, and let

that $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ is a \mathcal{F}_n -submartingale

i.e. $\mathbb{E}[X_n | \mathcal{F}_m] \geq X_m$ as when $n \geq m$ st. $X_m \geq 0$

$$E\left(\max_{0 \leq k \leq n} (X_k)^P\right) \leq \left(\frac{P}{P-1}\right)^P \cdot E[(X_n)^P]$$

§2.5 Martingale Convergence Theorem.

Suppose we have real numbers a_i , if $a_n, n=0, 1, 2, \dots$ st

$$a_0 \leq a_1 \leq \dots \leq a < \infty \text{ all } n.$$

Then $\lim_{n \rightarrow \infty} a_n$ exists and $\lim_{n \rightarrow \infty} a_n \leq a$

Thm 2.5.7: suppose that $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ is a sub-martingale such that $\sup_n E[X_n^+] < \infty$.

Then \exists a unique \mathcal{F}_∞ -meas. random variable X_∞ such that

$$E|X_\infty| < \infty, \lim_n X_n(w) = X_\infty(w) \text{ for p-a.s.}$$

Remark: Do we get $E[X_n] \rightarrow E(X_\infty)$?

Unfortunately NOT (see ex 2.5.8)

Recall Suppose f_n is a sequence of \mathcal{S} -meas. functions on (E, \mathcal{S}, μ) such that $f_n \rightarrow f$ a.e.

Do we have $\int f_n d\mu \rightarrow \int f d\mu$?

If \exists a \mathcal{S} -meas. function g st. $\int |g| d\mu < \infty$

and $|f_n| \leq g$ a.e.

(dominated convergence theorem)

Then $\int |f_n - f| d\mu \rightarrow 0$

If there exists some random variable Y st. $E[|Y|] < \infty$

and $|X_n| \leq |Y|$ a.s. (for each n) the LDC

gives $E[|X_n - X_\infty|] \rightarrow 0$

§2.6 Uniform Integrability

Prop. 2.6.1. Given a rv. X on (Ω, \mathcal{F}, P) st. $E[|X|] < \infty$

$P\{|w: |X(w)| > c\} \rightarrow 0$ as $c \rightarrow \infty$

$$\lim_{c \rightarrow \infty} \int_{\{|X| > c\}} |X| dP = 0$$

Defn. 2.6.2: An indexed family $\{X_\lambda, \lambda \in \Lambda\}$ of random variables on

(Ω, \mathcal{F}, P) is called uniformly integrable when

$$\limsup_{c \rightarrow \infty} \left[\int_{\{\mid X_\lambda \mid > c\}} \mid X_\lambda \mid dP \right] = 0$$

Thm 2.6.10 (Vitali) suppose that X and $\{X_n, n=0, 1, 2, \dots\}$ are RV's on (Ω, \mathcal{F}, P) st.

(a) $\{X_n, n=0, 1, 2, \dots\}$ is uniformly integrable (UI).

(b) $\lim_{n \rightarrow \infty} X_n(w) = X(w)$ P-a.s.

Then $E|X| < \infty$ and

$$E[|X_n - X|] \rightarrow 0 \text{ as } n \rightarrow \infty$$

In particular, $E[X_n] \rightarrow E[X]$

Thm 2.6.5 Given an indexed set of random variables $\{X_\lambda, \lambda \in \Lambda\}$ on (Ω, \mathcal{F}, P)

(a) If \exists an integrable RV. Γ st. $|X_\lambda| \leq \Gamma$ a.s.

then $\{X_\lambda, \lambda \in \Lambda\}$ is UI

(b) If for some constant $P \in (1, \infty)$, we have

$\sup_{\lambda} E[|X_\lambda|^P] < \infty$, then $\{X_\lambda, \lambda \in \Lambda\}$ is UI

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Defn. 2.6.2 A given set of RV's $\{X_\lambda, \lambda \in \Lambda\}$ on (Ω, \mathcal{F}, P) is uniformly integrable when

$$\lim_{c \rightarrow \infty} \left[\sup_{\lambda} \int_{\{\mid X_\lambda \mid > c\}} \mid X_\lambda \mid dP \right] = 0$$

Thm. 2.6.10. Suppose that $\{X_n, n=1, 2, \dots\}$ is UI, and $X_n \rightarrow X$, then

(a) $E|X| < \infty$

$$(b) \lim E[|X_n - X|] = 0$$

Thm. 2.6.5 Let $\{X_\lambda : \lambda \in \Lambda\}$ be a set RV's on (Ω, \mathcal{F}, P) .

(a) If \exists an integrable RV. Γ st. $|X_\lambda| \leq \Gamma$ a.s. then

$\{X_\lambda, \lambda \in \Lambda\}$ is UI.

(b) If for some constant $P \in (1, \infty)$, we have $E[|X_\lambda|^P] < \infty$

$$\sup_{\lambda}$$

then $\{X_\lambda\}$ is u.i.

Thm. 2.6.7. [Doob] Given an integrable random variable X on (Ω, \mathcal{F}, P) and any collection $\{\mathcal{G}_\lambda, \lambda \in \Lambda\}$, $\mathcal{G}_\lambda \subset \mathcal{F}$ of sub- σ -algs.

Let $Y_\lambda \stackrel{\Delta}{=} \mathbb{E}[X | \mathcal{G}_\lambda], \lambda \in \Lambda$.

then $\{Y_\lambda\}$ is. u.i.

Recall Thm. 2.5.7: suppose that $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ is a

Submartingale, st. $\sup_n \mathbb{E}[X_n^+] < \infty$

Then, \exists some \mathcal{F}_∞ -meas. rv. X_∞ st. $\mathbb{E}|X_\infty| < \infty$

and $\lim_{n \rightarrow \infty} X_n = X_\infty$ P-a.s.

In general, we don't get $\mathbb{E}[(X_n - X_\infty)] \rightarrow 0$. (convergence)

see e.g. 2.5.9

§ 2.7 Uniformly Integrable Supermartingales.

Thm 2.7.1 (a) Let $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ be a u.i. supermartingale.

(i.e. $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ is a supermartingale, and

$\{X_n, n=0, 1, 2, \dots\}$ is u.i.) Then there exists \mathcal{F}_∞ -meas. rv. X_∞

st. $\mathbb{E}|X_\infty| < \infty$, and

$$\lim_{n \rightarrow \infty} X_n = X_\infty \text{ a.s.} \quad \text{--- (1)}$$

$$\lim_{n \rightarrow \infty} \mathbb{E}[|X_n - X_\infty|] = 0 \quad \text{--- (2)}$$

$$X_n \geq \mathbb{E}[X_\infty | \mathcal{F}_n] \text{ for all } n=0, 1, 2, \dots \quad \text{--- (3)}$$

i.e. $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots + \infty\}$ is a closed supermartingale.

(b) Let $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ be a u.i. martingale. Then all statements of part (a) continue to hold, except that we can strengthen (3) to the following.

$$X_n = \mathbb{E}[X_\infty | \mathcal{F}_n] \text{ for all } n=0, 1, 2, \dots$$

Read proof.

Thm 2.7.4. Fix some $p \in (1, \infty)$, and suppose that

$\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ is a martingale such that

$\sup_n \mathbb{E}[|X_n|^p] < \infty$, i.e. the martingale is L_p -bounded for some $p \in (1, \infty)$.

Then there exists some \mathcal{F}_∞ -meas. r.v. X_∞ s.t. $E[\underbrace{|X_\infty|^P}_{\text{i.e., } X_\infty \text{ is } P\text{-th order}}] < \infty$
and
 $\lim_{n \rightarrow \infty} X_n = X_\infty$ P-a.s.

$$\lim_{n \rightarrow \infty} E[|X_n - X_\infty|^P] = 0$$

$$X_n = E[X_\infty | \mathcal{F}_n], n=0, 1, 2, \dots$$

$$\text{Let us take } E[\sup_n |X_n|^P] \leq (\frac{P}{P-1})^P E[|X_\infty|^P]$$

Proof: please read.

Thm. 2.7.6. (Lévy) Given an integrable rv. X . and a filtration.

$$\{f_n, n=0, 1, 2, \dots\} \text{ in } (\Omega, \mathcal{F}, P)$$

$$\lim_{n \rightarrow \infty} E[X | \mathcal{F}_n] = E[X | \mathcal{F}_\infty] \text{ a.s.}$$

$$\lim_n E\{|E[X | \mathcal{F}_n] - E[X | \mathcal{F}_\infty]| \} = 0$$

read proof. quite worth reading.

CH 3 Continuous Parameter Stochastic Processes

Defn. 3.1.1. A continuous parameter stochastic process on a probability space (Ω, \mathcal{F}, P) is a collection $\{X_t, t \in [0, \infty)\}$ of r.v.s.

i.e. each X_t is a \mathcal{F} -meas. \mathbb{R} -valued rv.)

Suppose that $\{x_t, t \in [0, +\infty)\}$ is a given stochastic process on (Ω, \mathcal{F}, P) , then for each $w \in \Omega$, we have a mapping

$$t \mapsto X_t(w) : [0, \infty) \rightarrow \mathbb{R}$$

This gives the sample path of the process corresponding to w .

A process is called continuous [right-continuous, left-continuous] when each and every sample path is continuous [right continuous, left continuous].

Defn. 3.1.6. Two stochastic processes $\{X_t, t \in [0, \infty)\}$ and

$\{Y_t, t \in [0, \infty)\}$ on a common prob. space (Ω, \mathcal{F}, P) are

indistinguishable when, for

$$A \stackrel{\Delta}{=} \{w : X_t(w) = Y_t(w) \text{ } t \in [0, \infty)\}$$

(i) $A \in \mathcal{F}$.

(ii) $P(A) = 1$

Defn. 3.1.8. Two stochastic processes $\{X_t\}$ and $\{Y_t\}$ are modifications of each other when

$$P\{\omega: X_t(\omega) = Y_t(\omega)\} = 1 \text{ for all } t \in [0, \infty)$$

(clearly, if $\{X_t\}$ and $\{Y_t\}$ are indistinguishable, then they are modifications of each other, however, the converse is generally false.)

— see 3.1.9 (please read)

Prop. 3.1.11. Suppose that $\{X_t\}$ and $\{Y_t\}$ on the same prob. space.

(Ω, \mathcal{F}, P) are modifications of each other.

If $\{X_t\}$ and $\{Y_t\}$ are either r.c. or l.c. then

they are indistinguishable.

Proof: suppose that $\{X_t\}$ and $\{Y_t\}$ are r.c.

Fix the set $\{t_n, n=1, 2, \dots\}$ of all non-negative rationals.

Put $\mathcal{S}^* = \bigcap_n \{\omega: X_{t_n}(\omega) = Y_{t_n}(\omega)\}$

Now $\{\omega: X_{t_n}(\omega) = Y_{t_n}(\omega)\} \in \mathcal{F}$

Since X_{t_n} and Y_{t_n} are \mathcal{F} -meas.

$\therefore \mathcal{S}^* \in \mathcal{F}$.

Moreover, $P(X_{t_n} = Y_{t_n}) = 1$ all $n=1, 2, \dots$

Thus

$$P((\mathcal{S}^*)^c) = P(\bigcup \{\omega: X_{t_n} \neq Y_{t_n}\})$$

$$\leq \sum_n P[X_{t_n} \neq Y_{t_n}] = 0$$

$$\text{Therefore } P(\mathcal{S}^*)^c = 0 \Rightarrow P(\mathcal{S}^*) = 1$$

Fix some $\omega \in \mathcal{S}^*$, then

$$X_{t_n}(\omega) = Y_{t_n}(\omega) \quad \text{all } n=1, 2, \dots$$

Fix arbitrary $t \in [0, \infty)$ Thus there exists some sequence

$\{s_k, k=1, 2, \dots\}$ of rationals. st

$$\lim_k s_k = t \quad \text{and } t \leq s_{k+1} \leq s_k$$

rationals are dense in the real line

also

but $X_{Sk}(w) = Y_{Sk}(w)$ all k .

since X , and Y are r.c., we have

$$\lim_{k \rightarrow \infty} X_{Sk}(w) = x_t(w) \quad \text{and} \quad \lim_{k \rightarrow \infty} Y_{Sk}(w) = y_t(w)$$

$$\therefore x_t(w) = y_t(w)$$

Feb 4, 2009 STAT902. Probability Theory II.

Recall that a continuous parameter stochastic process is an indexed collection $\{X_t, t \in [0, \infty)\}$ on (Ω, \mathcal{F}, P)

i.e. $X_t : \Omega \rightarrow \mathbb{R}$ is \mathcal{F} -meas. for each t .

We can regard this as a mapping

$$X : [0, \infty) \times \Omega \rightarrow \mathbb{R}$$

Define the product σ -alg $\mathcal{B}([0, \infty) \otimes \mathcal{F}$ on $[0, \infty) \otimes \Omega$, Then the process $\{X_t\}$ is said to be jointly measurable when

$$X : [0, \infty) \otimes \Omega \rightarrow \mathbb{R} \text{ is } (\mathcal{B}([0, \infty) \otimes \mathcal{F})\text{-meas.}$$

In particular, $w \mapsto X_t(w) : \Omega \rightarrow \mathbb{R}$ is \mathcal{F} -meas.

$t \mapsto X_t(w) : [0, \infty) \rightarrow \mathbb{R}$, is $\mathcal{B}([0, \infty))$ -meas.

Suppose also $X \geq 0$ Fubini - .

$$\int_{\Omega} \left\{ \int_0^{\infty} X_t(w) dt \right\} dP(w) = \int_0^{\infty} \left\{ \int_{\Omega} X_t(w) dP(w) \right\} dt$$

$$\text{i.e. } \mathbb{E} \left[\int_0^{\infty} X_t dt \right] = \int_0^{\infty} \mathbb{E}[X_t] dt$$

Def. 3.1.4 A filtration in (Ω, \mathcal{F}, P) is a collection of σ -algebras $\{\mathcal{F}_t, t \in [0, \infty)\}$ st.

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \text{ where } 0 \leq s < t < \infty$$

A process $\{X_t, t \in [0, \infty)\}$ is called $\{\mathcal{F}_t\}$ adapted when

X_t is \mathcal{F}_t -meas for each $t \in [0, \infty)$

Remark 3.1.5 $\mathcal{F}_{\infty} \triangleq \sigma\{\mathcal{F}_t, t \in [0, \infty)\}$

Basic question suppose that $\{(X_t, \mathcal{F}_t), t \in [0, \infty)\}$ is an adapted process, and

$\{Y_t, t \in [0, \infty)\}$ is a modification of $\{X_t\}$

i.e. $P\{Y_t \neq X_t\} = 0$ for all t

Fe.

Is $\{(y_t, \mathcal{F}_t)\}$ adapted?

Not necessarily, since the event $\{y_t \neq x_t\}$ need not be a member of \mathcal{F}_t (although of course $\{y_t \neq x_t\} \subset \mathcal{F}$)

Put $\mathcal{N} = \{N \in \mathcal{F}: P(N) < 0\}$

then $\{x_t \neq y_t\} \in \mathcal{N}$ for all $t \in [0, \infty)$

Suppose the filtration $\{\mathcal{F}_t\}$ is "well-behaved" in the sense that

$$\mathcal{N} \subset \mathcal{F}_0 \subset \mathcal{F}_t$$

$\mathcal{N} \subset \mathcal{F}_t$ for all $t \in [0, \infty)$

In particular, $\{x_t \neq y_t\} \in \mathcal{F}_t$ for each t

From this we can conclude that y_t is \mathcal{F}_t -meas for each $t \in [0, \infty)$.

i.e. $\{(y_t, \mathcal{F}_t)\}$ is adapted.

see Prop. 3.1.18 for the details.

Basic Question: suppose that $\{(x_t, \mathcal{F}_t)\}$ is an adapted process and suppose $X \geq 0$.

Put $Y_t(w) = \int_{[0,t]} X_s(w) ds$ for all $t \in [0, \infty)$

Is $\{(Y_t, \mathcal{F}_t)\}$ an adapted process?

$$f: (\mathbb{E}_1, \mathcal{S}_1, \mathcal{M}_1) \otimes (\mathbb{E}_2, \mathcal{S}_2, \mathcal{M}_2) \rightarrow [0, \infty]$$

$$x_1 \mapsto \int_{\mathbb{E}_2} f(x_1, x_2) d\mathcal{M}_2(x_2) \text{ is } \mathcal{S}_1\text{-meas.}$$

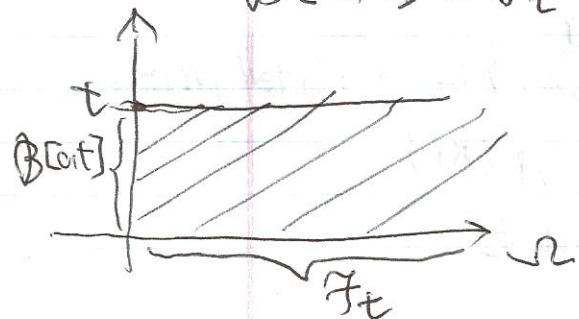
Defn. 3.1.21.

Suppose that $\{X_t\}$ is a process on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}$ is a filtration in $(\Omega, \mathcal{F}, \mathbb{P})$

then $\{X_t\}$ is called \mathcal{F}_t -progressively measurable when, for each $t \in [0, \infty)$

the mapping $(s, w) \mapsto X_s(w): [0, t] \times \Omega \rightarrow \mathbb{R}$

is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ -meas.



Suppose that $\{X_t\}$ is $\{\mathcal{F}_t\}$ -progr. meas. and $X \geq 0$

put $Y_t(w) = \int_{[0,t]} X_s(w) dS$, $w \in \Omega$, $t \in [0, \infty)$

Then for each $t \in [0, \infty)$, see from 1.2.40 that Y_t is \mathcal{F}_t -meas.

i.e. $\{(Y_t, \mathcal{F}_t)\}$ is adapted.

If $\{X_t\}$ is $\{\mathcal{F}_t\}$ -progressively meas. then it is immediate from the defn.

that (i) $\{X_t\}$ is \mathcal{F}_t -adapted

(ii) $\{X_t\}$ is jointly meas.

Prop. 3.1.25 suppose that $\{(X_t, \mathcal{F}_t)\}$ is an adapted process, and that

$\{X_t\}$ is r.c (or L.C), then

$\{X_t\}$ is \mathcal{F}_t -PM.

Proof: Very simple!

Def. 3.1.26 : Let $\{\mathcal{F}_t\}$ be a given filtration in (Ω, \mathcal{F}, P)

For each $t \in [0, \infty)$, put

$$\mathcal{F}_{t+} \triangleq \bigcap_{s > t} \mathcal{F}_s$$

Then $\{\mathcal{F}_{t+}, t \in [0, \infty)\}$ is the right continuous enlargement of

$\{\mathcal{F}_t, t \in [0, \infty)\}$ and $\{\mathcal{F}_t\}$ is called a right continuous filtration

when $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \in [0, \infty)$

§ 3.2. THE WIENER PROCESS (a Brownian Motion)

Defn. 3.2.1 A process $\{X_t\}$ is said has independent increments when for every choice of instants, $0 = t_0 < t_1 < t_2 \dots < t_n < \infty$ the resulting set of RV's

$\{X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}\}$ is independent.

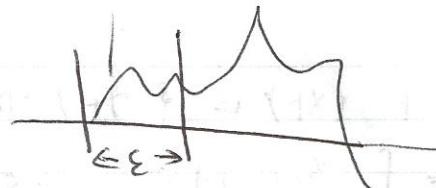
Defn. 3.2.2. A scalar Wiener process $\{W_t, t \in [0, \infty)\}$ is an \mathbb{R} -valued with the following properties :

(i) $W_0 \equiv 0$

(ii) $t \mapsto W_t(w)$ are continuous for every w cauchy

(iii) for $0 \leq s < t$, the increment $W_t - W_s$ is Gaussian distributed with zero mean and variance $t-s$, i.e. $W_t - W_s \sim N(0, t-s)$

(IV) $\{W_t\}$ has independent increments.



Defn. 3.2.11. A scalar Wiener process is an adapted process $\{(W_t, \mathcal{F}_t), t \in [0, \infty)\}$ in which W_t is \mathbb{R} -valued with the following properties:

- (i) $W_0 \equiv 0$
- (ii) $t \mapsto W_t(w)$ are continuous for each w
- (iii) $W_t - W_s \sim N(0, t-s)$, $0 \leq s < t$
- (iv) when $0 \leq s < t$ then $(W_t - W_s) \perp \mathcal{F}_s$.

$$0 \quad \mathcal{F}_s \quad s \quad \sqrt{t-s} \quad t$$

One can show that a Wiener process in the sense of Defn 3.2.11 is also a Wiener process in the sense of Defn. 3.2.2., in particular $\{W_t\}$ has independent increments.

Feb 9, 2009 STAT 902.

Defn 3.2.11. An \mathbb{R}^d -valued process $\{(W_t, \mathcal{F}_t), t \in [0, \infty)\}$ on (Ω, \mathcal{F}, P) is a scalar Wiener process (or Brownian Motion)

when (i) $W_0 \equiv 0$

(ii') $t \mapsto W_t(w)$ is continuous for each w .

(iii') $W_t - W_s \sim N(0, t-s)$, $0 \leq s < t < \infty$. Multi-dimensional,
 $\sim N(0, (t-s)I_d)$

(iv) $W_t - W_s \perp \mathcal{F}_s$, $0 \leq s < t < \infty$

§ 3.3 stopping times:

Defn: 3.3.1. Given a filtration $\{\mathcal{F}_t, t \in [0, \infty)\}$ in (Ω, \mathcal{F}, P) , then a \mathcal{F} -meas. mapping $T: \Omega \rightarrow [0, \infty]$ is random time.

A random time $T: \Omega \rightarrow [0, \infty]$ is a $\{\mathcal{F}_t\}$ -stopping time when

$$\{T \geq t\} \in \mathcal{F}_t \quad t \in [0, \infty)$$

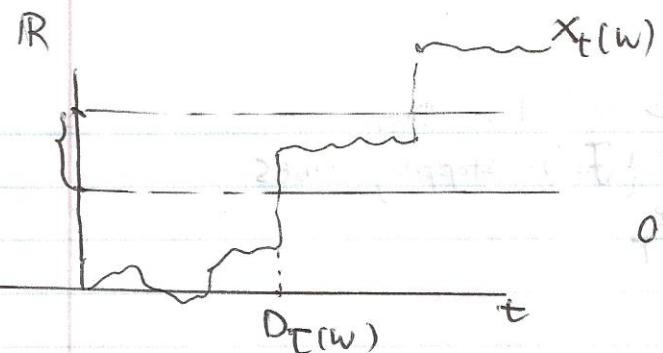
and is a $\{\mathcal{F}_t\}$ -optional time when

$$\{T < t\} \in \mathcal{F}_t \quad t \in [0, \infty)$$

Remark 3.3.6. Fix $I \subset \mathbb{R}$ and suppose that $\{X_t\}$ is \mathbb{R} -valued process

$$D_I(w) = \inf \{t \in [0, \infty): X_t(w) \in I\} \quad \text{all } w \in \Omega$$

(note $\inf \{\emptyset\} = +\infty$)

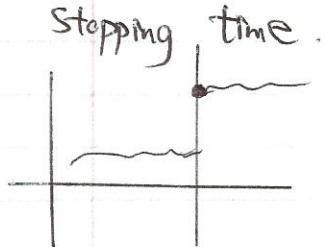


41.

First entrance time of C by $\{X_t\}$
or début of C by X_t

prop. 3.3.7 Given a non-empty set $C \subset \mathbb{R}$ and an adapted process $\{(X_t, \mathcal{F}_t), t \in [0, \infty)\}$ Then

(b) If C is closed, and $\{X_t\}$ is continuous, then D_C is a $\{\mathcal{F}_t\}$ -stopping time.



Konita + Watanabe

RC. also holds. proof (very advanced)

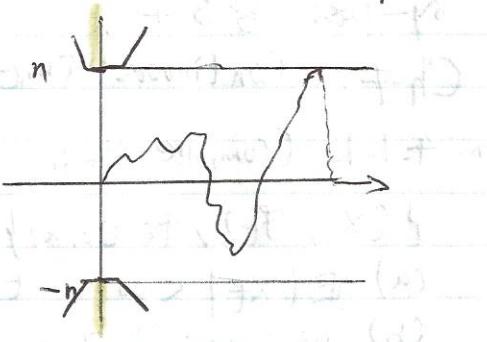
Remark 3.3.8. suppose $\{(X_t, \mathcal{F}_t), t \in [0, \infty)\}$ is a continuous adapted process, for each $n = 1, 2, 3, \dots$

$$T_n(w) \triangleq \inf \{t \in [0, \infty) : |X_t(w)| \geq n\}$$

Clearly $T_n(w) = D_{C_n}(w)$ where

$$C = \{x \in \mathbb{R} / |x| \geq n\}$$

\uparrow
closed.



Therefore, each T_n is a $\{\mathcal{F}_t\}$ -stopping time.

Definition 3.3.11 (compare Defn. 2.1.10)

Given a $\{\mathcal{F}_t\}$ -stopping time T , Define $\mathcal{F}_T \triangleq \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t\}$ for $t \in [0, \infty]$.

We call \mathcal{F}_T the pre- σ -algebra of T .

Prop. 3.3.12. (compare prop. 2.1.11)

Given a $\{\mathcal{F}_t\}$ -stopping time T , Then

(a) \mathcal{F}_T is a σ -alg. and $\mathcal{F}_T \subset \mathcal{F}_\infty$

(b) T is \mathcal{F}_T -meas. thus $\sigma\{T\} \subset \mathcal{F}_T$

(c) If $T(w) = t$ for some $t \in [0, \infty]$, then $\mathcal{F}_T = \mathcal{F}_t$

Prop. 3.3.13. (compare prop. 2.1.12)

Given the $\{\mathcal{F}_t\}$ -stopping times S and T , then

(a) $SAT, SVT, S+T$ are $\{\mathcal{F}_t\}$ -stopping times.

(b) if $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$

(c) $\mathcal{F}_{SAT} = \mathcal{F}_S \cap \mathcal{F}_T$

(d) $\{S < T\}, \{S = T\}, \{S > T\}$ are members of \mathcal{F}_{SAT} .

(e) if $A \in \mathcal{F}_S$, then $A \cap \{S \leq T\}, A \cap \{S < T\}, A \cap \{S = T\}$
are members of \mathcal{F}_{SAT} .

(But $A \cap \{S \geq T\} \notin \mathcal{F}_{SAT}$) in general.

$\mathcal{F}_T = \{A \text{ s.s.}: A \cap \{T=n\} \in \mathcal{F}_n, n=0, 1, 2, \dots, +\infty\}$

$\mathcal{F}_T = \{A \text{ s.s.}: A \cap \{T=t\} \in \mathcal{F}_t, t \in [0, \infty]\}$

$$\nearrow \quad \{T \leq n\} = \bigcup_{k=0}^n \{T=k\} \quad \{T \leq t\} = \bigcup_{0 \leq s \leq t} \{T=s\}$$

By-pass § 3.4.

Ch 4. Continuous time/parameter Martingale

Defn 4.1.1. (Compare Defn. 2.2.1) an \mathbb{R} -valued adapted process

$\{(X_t, \mathcal{F}_t), t \in [0, \infty)\}$ is a supermartingale when

(a) $\mathbb{E}[X_t] < \infty \quad t \in [0, +\infty)$

(b) $\mathbb{E}[X_t | \mathcal{F}_s] \leq X_s \quad \text{a.s. when } 0 \leq s \leq t < \infty$

Bypass Ex. 4.1.3.

Remark 4.1.4. Given $p \in (1, \infty)$, a supermartingale $\{(X_t, \mathcal{F}_t), t \in [0, \infty)\}$ is an L_p -supermartingale when $\mathbb{E}[|X_t|^p] < \infty$ for all $t \in [0, +\infty)$ and is an L_p -bounded supermartingale.

when $\sup_{t \geq 0} \mathbb{E}[|X_t|^p] < \infty$

Remark 4.1.6. Given a filtration $\{\mathcal{F}_t\}$ in $(\Omega, \mathcal{F}, \mathbb{P})$

(a) $M(\{\mathcal{F}_t\}, P)$ denotes the set of all processes $\{\mathcal{X}_t\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$

st. $\{(X_t, \mathcal{F}_t)\}$ is a martingale.

$\mathbb{M}^c(\{\mathcal{F}_t\}, P)$ denotes the set of all continuous processes $\{X_t\}$ on (Ω, \mathcal{F}, P) such that $\{(X_t, \mathcal{F}_t)\}$ is a martingale.

(b) when $P \in (1, \infty)$, then

$\mathbb{M}_P(\{\mathcal{F}_t\}, P)$ denotes the set of all processes $\{X_t\}$ on (Ω, \mathcal{F}, P)

st. (X_t, \mathcal{F}_t) is an L_p -martingale

$\mathbb{M}_{p,b}(\{\mathcal{F}_t\}, P)$ denotes the set of all processes $\{X_t\}$ on (Ω, \mathcal{F}, P)

st. $\{(X_t, \mathcal{F}_t)\}$ is an L_p -bounded martingale.

Prop. 4.1.8. Fix scalar wiener process $\{(W_t, \mathcal{F}_t) : t \in [0, +\infty)\}$ on (Ω, \mathcal{F}, P) then (a) $\{(W_t, \mathcal{F}_t)\}$ is an L_2 -martingale

(b) for $X_t \triangleq W_t^2 - t$, the process $\{(X_t, \mathcal{F}_t)\}$ is martingale.

(a) $W_t = W_t - W_0 \sim N(0, t) \quad t \in [0, +\infty)$

$$\mathbb{E}[W_t]^2 = t, \quad t \in [0, +\infty)$$

$$\text{i.e. } \mathbb{E}|W_t| < \infty \quad t \in [0, +\infty)$$

fix $0 \leq s < t < \infty$. then $W_t - W_s \perp \mathcal{F}_s$.

$$\text{i.e. } \mathbb{E}[W_t - W_s | \mathcal{F}_s] = \mathbb{E}[W_t - W_s] = 0$$

$$\text{i.e. } \mathbb{E}[W_t - W_s | \mathcal{F}_s] = 0$$

$$\mathbb{E}[W_t | \mathcal{F}_s] - \underbrace{\mathbb{E}[W_s | \mathcal{F}_s]}_{W_s} = 0$$

$$\therefore \mathbb{E}[W_t | \mathcal{F}_s] = W_s$$

Note that $\{(W_t, \mathcal{F}_t)\}$ is an L_2 -martingale. (re $w \in \mathbb{M}_2(\{\mathcal{F}_t\}, P)$)

but $w \notin \mathbb{M}_{2,b}(\{\mathcal{F}_t\}, P)$.

(b) fix $0 \leq s < t < \infty$

$$\mathbb{E}[|X_t|^2] = \mathbb{E}[|W_t^2 - t|^2] \leq \mathbb{E}[|W_t|^2] + t < \infty$$

$$\mathbb{E}[W_s W_t | \mathcal{F}_s] = W_s \mathbb{E}[W_t | \mathcal{F}_s] = W_s^2 \quad \text{--- (1)}$$

$$\mathbb{E}[(W_t - W_s)^2 | \mathcal{F}_s] = \mathbb{E}[W_t^2 - 2W_s W_t + W_s^2 | \mathcal{F}_s]$$

$$= \mathbb{E}[W_t^2 + W_s^2 | \mathcal{F}_s] - 2 \mathbb{E}[W_s W_t | \mathcal{F}_s] =$$

by ①

$$= \mathbb{E}[w_t^2 + w_s^2 | \mathcal{F}_s] - 2w_s^2$$

$$= \mathbb{E}[w_t^2 - w_s^2 | \mathcal{F}_s] \quad \text{--- ②}$$

Now $(w_t - w_s) \sim N(0, t-s)$ and $(w_t - w_s) \perp\!\!\!\perp \mathcal{F}_s$

$$\therefore \mathbb{E}[(w_t - w_s)^2 | \mathcal{F}_s] = \mathbb{E}[(w_t - w_s)^2] = t-s \quad \text{--- ③}$$

From ② and ③ $\mathbb{E}[w_t^2 - w_s^2 | \mathcal{F}_s] = t-s$

$$\mathbb{E}[w_t^2 | \mathcal{F}_s] - \underbrace{\mathbb{E}[w_s^2 | \mathcal{F}_s]}_{w_s^2} = t-s$$

$$\therefore \mathbb{E}[w_t^2 | \mathcal{F}_s] - w_s^2 = t-s$$

$$\Rightarrow \mathbb{E}[w_t^2 | \mathcal{F}_s] - t = (w_s^2 - s)$$

$$\therefore \underbrace{\mathbb{E}[w_t^2 - t | \mathcal{F}_s]}_{xt} = \underbrace{w_s^2 - s}_{xs}$$

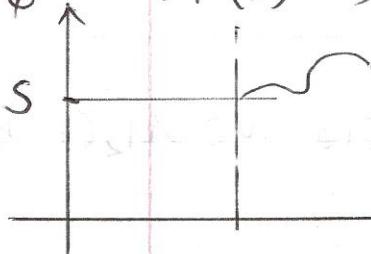
Feb 11, 2009 STAT 902 Theory of probability I.

§4.2. Sample Path Properties.

Given a function $\varphi: [0, \infty) \rightarrow \mathbb{R}$

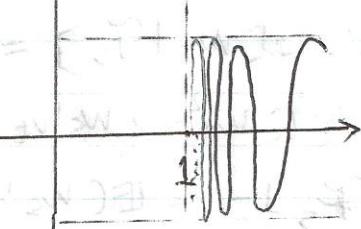
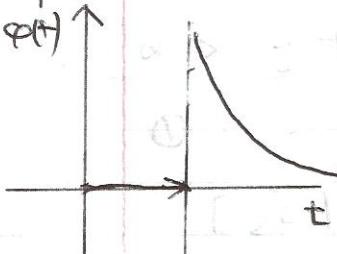
Then $\varphi(\cdot)$ has a finite right limit at some $t \in [0, \infty)$ when there exists some $\zeta \in \mathbb{R}$. st. for each $\varepsilon > 0$, \exists some $\delta(\varepsilon) > 0$

st. $|\varphi(s) - \zeta| < \varepsilon$ for $s \in (t, t + \delta(\varepsilon))$



ex. $\varphi(t) \triangleq \begin{cases} 0 & 0 \leq t \leq 1 \\ \frac{1}{t-1} & t > 1 \end{cases}$

ex. $\varphi(t) \triangleq \begin{cases} 0 & 0 \leq t \leq 1 \\ s_n(\frac{1}{t-1}) & t \geq 1 \end{cases}$

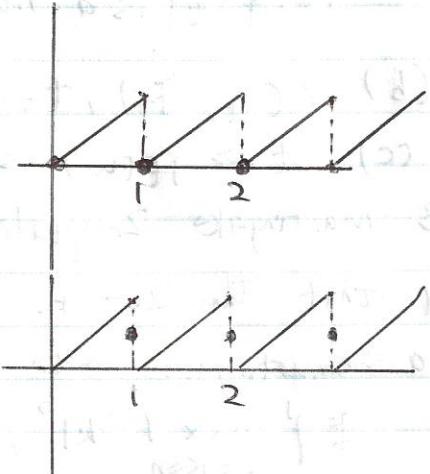
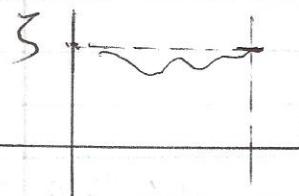


Denote ζ by $\varphi(t+)$. Then f_n is right continuous when

- (a) φ has a finite right limit $\varphi(t+)$ at each $t \in [0, \infty)$

(b) $\varphi(t) = \varphi(t+)$ $t \in [0, \infty)$

In the same way we can formulate the notion of a finite left limit $\varphi(t-)$ for $t \in (0, \infty)$



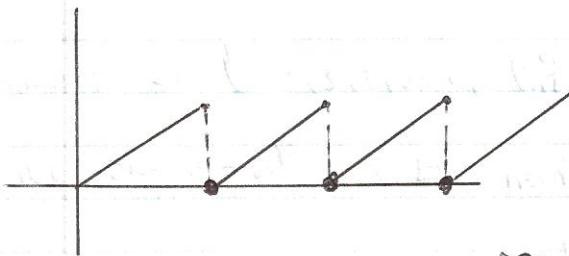
Remark 4.2.4. A function $\varphi: [0, \infty) \rightarrow \mathbb{R}$ is a **cadlag function** (continuous on the right, finite limits on left), **Cadlag function** or **Skorokhod functions**. When

- (a) φ is right-continuous on $[0, \infty)$

- (b) φ has finite left limit $\varphi(t-)$ for each $t \in (0, \infty)$

$$\mathbb{E}[x_t | \mathcal{F}_s] = x_s \text{ a.s.}$$

$$t \rightarrow x_{\lfloor t \rfloor}$$



Recall Def 3.4.2: A filtration $\{\mathcal{F}_t, t \in [0, \infty)\}$ in $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard filtration (or satisfies the usual conditions)

- (a) every \mathbb{P} -null events of \mathcal{F} is a member of \mathcal{F}_0 (i.e. of each \mathcal{F}_t)
- (b) $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \in [0, \infty)$ where

$$\mathcal{F}_{t+} = \bigcap_{s > t} \mathcal{F}_s \quad \mathcal{F}_t \subset \mathcal{F}_{t+} \text{ in general right continuity}$$

corollary 4.2.19

suppose that $\{\mathcal{F}_t\}$ is a standard filtration in $(\Omega, \mathcal{F}, \mathbb{P})$, if (X_t, \mathcal{F}_t) $t \in [0, \infty)$ is a martingale. then there exists an $\{\mathcal{F}_t\}$ adapted process $\{Y_t, t \in [0, \infty)\}$ such that

(a) $P[X_t = Y_t] = 1$ for all $t \in [0, \infty)$

i.e. $\{Y_t\}$ is a modification of $\{X_t\}$

(b) $\{(Y_t, \mathcal{F}_t), t \in [0, \infty)\}$ is a martingale.

(c) $t \mapsto Y_t(w)$ is contol for each w .

§ 4.3 Martingale Inequalities:

Recall that Thm 2.4.4. Take $p \in (1, \infty)$, if $\{(X_n, \mathcal{F}_n), n=0,1,2,\dots\}$ is a submartingale with $X_n \geq 0$, then

$$\mathbb{E} \left\{ \max_{0 \leq k \leq n} |X_k|^p \right\} \leq \left(\frac{p}{p-1} \right)^p \mathbb{E}[|X_n|^p]$$

Thm 4.3.3. (ii) Take $p \in (1, \infty)$ if $\{(X_t, \mathcal{F}_t), t \in [0, \infty)\}$ is a submartingale (right-continuous) with $X_t \geq 0$

(i.e. each sample path of X_t is a rc. function) then

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq T} |X_t|^p \right\} \leq \left(\frac{p}{p-1} \right)^p \mathbb{E}[|X_T|^p]$$

§ 4.4 Martingale Convergence Theorem:

Recall Thm 2.5.7

Suppose that $\{(X_n, \mathcal{F}_n), n=0,1,2,\dots\}$ is a submartingale.

$\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|] < \infty$, then \exists an \mathcal{F}_∞ -meas random variable X_∞ st. $\mathbb{E}|X_\infty| < \infty$ and

$$\lim_{n \rightarrow \infty} X_n(w) = X_\infty(w) \text{ for } P\text{-as. } w.$$

Thm 4.4.7. Suppose that $\{(X_t, \mathcal{F}_t), t \in [0, \infty)\}$ is a right-continuous submartingale, such that

$\sup_{t \in [0, \infty)} \mathbb{E}[|X_t|] < \infty$ Then there exists an \mathcal{F}_∞ -meas.

random variable X_∞ st. $\mathbb{E}|X_\infty| < \infty$ and

$$\lim_{t \rightarrow \infty} X_t(w) = X_\infty(w) \text{ for } P\text{-a.a. } w$$

Recall Defn 2.3.4: a closed

The index collection $\{(X_n, \mathcal{F}_n) | n=0, 1, 2, \dots\}$ is a closed supermartingale when

- (a) $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots\}$ is a supermartingale.
- (b) $\mathcal{F}_\infty = \sigma\{\mathcal{F}_n, n=0, 1, \dots\}$
- (c) X_∞ is \mathcal{F}_∞ -meas. and it is integrable.
 $E[X_\infty] < \infty$ & $E[X_\infty | \mathcal{F}_n] \leq X_n$ a.s.

Defn 4.4.9.

The index collection $\{(X_t, \mathcal{F}_t) | t \in [0, \infty]\}$ is a closed supermartingale when

- (a) $\{(X_t, \mathcal{F}_t), t \in [0, \infty)\}$ is a supermartingale
- (b) $\mathcal{F}_\infty = \sigma\{\mathcal{F}_t, t \in [0, \infty)\}$
- (c) X_∞ is \mathcal{F}_∞ -meas st.
 $E[X_\infty] < \infty$ & $E[X_\infty | \mathcal{F}_t] \leq X_t$ a.s.

Recall Thm 2.7.1(b) suppose $\{(X_n, \mathcal{F}_n) | n=0, 1, 2, \dots\}$ is a UI martingale on (Ω, \mathcal{F}, P) , then there exists some \mathcal{F}_∞ -meas RV. X_∞ st. $E[X_\infty] < \infty$ and $\lim_{n \rightarrow \infty} X_n = X_\infty$ a.s.

$$\lim_n E[X_n - X_\infty] = 0$$

$\{(X_n, \mathcal{F}_n), n=0, \dots, \infty\}$ is a closed martingale.

$$X_n = E[X_\infty | \mathcal{F}_n] \text{ a.s. for all } n=0, 1, 2, \dots$$

Thm 4.4.11 (b) suppose that $\{(X_t, \mathcal{F}_t) | t \in [0, \infty]\}$ is a UI right-continuous martingale on (Ω, \mathcal{F}, P) then there exists some \mathcal{F}_∞ -meas. RV. X_∞ st. $E[X_\infty] < \infty$ and $\lim_{t \rightarrow \infty} X_t = X_\infty$ a.s.

$$\lim_{t \rightarrow \infty} E[|X_t - X_\infty|] = 0$$

$\{(X_t, \mathcal{F}_t) | t \in [0, \infty]\}$ is a closed martingale

$$X_t = E[X_\infty | \mathcal{F}_t] \text{ is a.s. for all } t \in [0, \infty)$$

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§4.5 O.S. Thm.

Recall Thm. 2.3.9 suppose that $\{(X_n, \mathcal{F}_n), n=0, 1, 2, \dots, +\infty\}$ is a closed supermartingale, and that S , and T are stopping times $S \leq T$. Then

$$(i) E|X_T| < \infty$$

$$(ii) E[X_T | \mathcal{F}_S] \leq X_S$$

pre-sigma-algebra

note

Thm. 4.5.4. suppose that $\{(X_t, \mathcal{F}_t), t \in [0, +\infty]\}$ is a r.c closed supermartingale [recall defn. 4.4.9] and $S \leq T$ are stopping times.

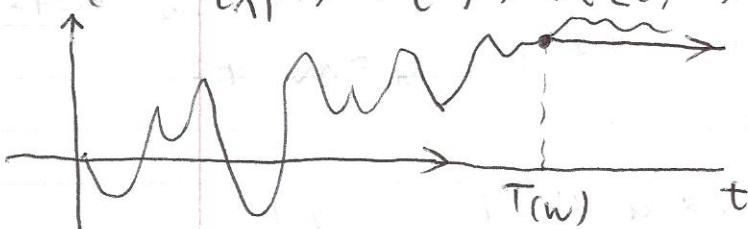
$S \leq T$, then

$$(a) E|X_T| < \infty$$

$$(b) E[X_T | \mathcal{F}_S] \leq X_S$$

Cor. 4.5.8 suppose that $\{(X_t, \mathcal{F}_t), t \in [0, +\infty)\}$ is a r.c martingale, and T is a stopping time. Then

$\{(X_{t \wedge T}, \mathcal{F}_t), t \in [0, \infty)\}$ is a martingale



$$X_{t \wedge T(w)} = \begin{cases} X_t(w) & 0 \leq t \leq T(w) \\ X_{T(w)}(w) & t > T(w) \end{cases}$$

$$\{T \leq t\} \in \mathcal{F}_t$$

§4.6 Local martingale

Defn: an \mathbb{R} -valued r.c adapted process $\{(X_t, \mathcal{F}_t) t \in [0, \infty)\}$ is a local martingale when there exists a sequence of stopping times $\{T_n, n=0, 1, 2, \dots\}$ st.

$$(a) 0 \leq T_n(w) \leq T_{n+1}(w)$$

$$(b) \lim_{n \rightarrow \infty} T_n = +\infty \text{ a.s.}$$

$$(c) \{(X_{t \wedge T_n}, \mathcal{F}_t), t \in [0, \infty)\}$$
 is a martingale for each $n=0, 1, 2, \dots$

continuous.

$\mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$

Remark 4.6.5 / fix a filtration $\{\mathcal{F}_t\}$ in (Ω, \mathcal{F}, P)
 write $\mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$ to denote the collection of all processes
 $\{X_t, t \in [0, \infty)\}$ st. $\{(X_t, \mathcal{F}_t), t \in [0, \infty)\}$ is a
 loc. martingale.

Remark: 4.6.6 $\mathcal{M}(\{\mathcal{F}_t\}, P) \subsetneq \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$

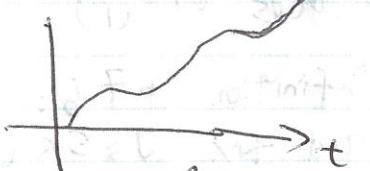
Prop. 4.6.7: Let $X \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$ if $X_t \geq 0$ for each $t \geq 0$
 then $\{(X_t, \mathcal{F}_t), t \in [0, \infty)\}$ is a supermartingale.

Read proof:

§4.7. The quadratic variation process: Deltachase - Meyer

Defn. Fix some $X \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$, a process $\{A(t), t \in [0, \infty)\}$
 is a quadratic variation process for the local martingale X when

- (a) $\{(A(t), \mathcal{F}(t)) : t \in [0, \infty)\}$ is a continuous adapted process null
 at $t=0$ (i.e. $A(0)=0$)
- (b) $t \mapsto A(t, \omega)$ is monotonically increasing
- (c) $X^2 A \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$



Condition 4.7.4. From now on, the filtration $\{\mathcal{F}_t, t \in [0, \infty)\}$ in (Ω, \mathcal{F}, P) is such that $\mathcal{N} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_t$

"collection of $\{N \in \mathcal{F}: P(N)=0\}$ "

Lemma 4.7.5. (Itô)

Given a local martingale $X \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$ st.

$|X(t, \omega)| \leq c < \infty$ for all (t, ω) "uniformly bounded"

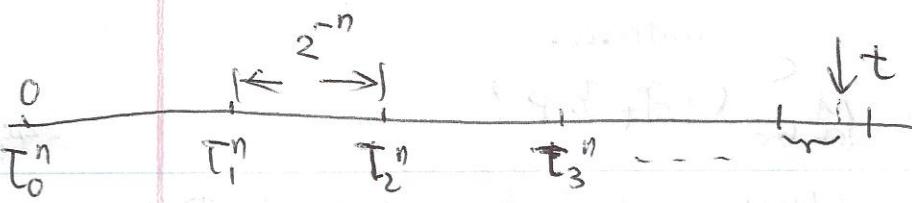
put $\underbrace{\tau_k^n = k 2^{-n}}_{\text{dyadic rationals}}$ all $n, k=0, 1, 2, \dots$

For each $n=0, 1, 2, \dots$ define $\{A_n(t), t \in [0, \infty)\}$ as follows:

$$A_n(t, \omega) \triangleq \sum_{0 \leq k \leq \infty} |X(t \wedge \tau_{k+1}^n, \omega) - X(t \wedge \tau_k^n, \omega)|$$

for all $t \in [0, \infty)$ $\omega \in \Omega$.

finitely many terms only
 for each $0 \leq t \leq \infty$



ECT 3142
Friday 27.
5:30pm

For each fixed $t \in [0, \infty)$, the sequence of r.v's.

$\{A_n(t), n=0, 1, 2, \dots\}$ converges in $L_2(\Omega, \mathcal{F}, P)$
to a random variable $A(t)$ s.t. $\{A(t)\}$ is a quadratic variation process for X .

i.e. for each $t \in [0, \infty)$, there exists a r.v. $A(t)$, s.t.

$$\mathbb{E}[|A(t)|^2] < \infty$$

$$\text{and } \mathbb{E}[|A_n(t) - A(t)|^2] \rightarrow 0 \text{ as } n \rightarrow \infty$$

Moreover, $\{A(t), t \in [0, \infty)\}$ is a q.v. process for X .

At this stage, we must want

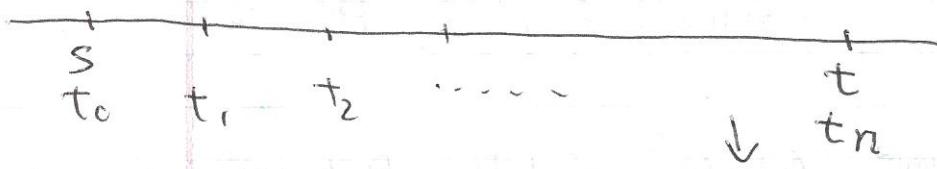
- (1) establish that the q.v. process $\{A(t)\}$ given by Lemma 4.7.5 is unique.
- (2) get rid of uniform boundedness in Lemma 4.7.5.

We focus on (1).

Definition 4.7.6: Take a function $A: [0, \infty) \rightarrow \mathbb{R}$.
and fix $0 \leq s < t < \infty$



Then A has bounded variation over the interval $[s, t]$



when there is a constant $C \in [0, \infty)$ s.t.

$$\sum_{k=1}^n |A(t_k) - A(t_{k-1})| < C \text{ for all finite partitions.}$$

$$s = t_0 < t_1 < t_2 < \dots < t_n = t$$

Put $V[A; s, t] = \sup \left\{ \sum_{k=1}^n |A(t_k) - A(t_{k-1})| \right\}$

where the supremum is over all finite partitions.

$$s = t_0 < t_1 < \dots < t_n = t$$

then A has finite variation over $[s, t]$ when

$$V[A; s, t] < \infty$$

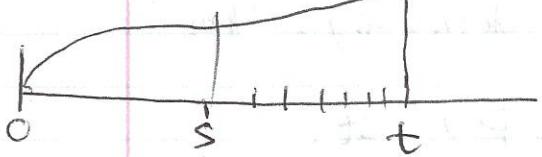
The function $\frac{A(t) - A(s)}{t - s}$ for all $t \in [s, \infty)$

$t \rightarrow V[A; s, t]$ for all $t \in [s, \infty)$ is called the total variation function of the function A .

The function A has locally bounded total variation when

$$V[A; s, t] < \infty \text{ for each } t \in [s, \infty)$$

Ex. $A: [0, \infty) \rightarrow \mathbb{R}$ is monotonically increasing.



$$V[A; s, t] = A(t) - A(s) < \infty$$

Variation of A over $[s, t]$

Feb 25, B-D STAT 902, 2009

$$V[A; s, t] = [0, \infty)$$

when $V[A; s, t] < \infty$ then A has finite variation over $[s, t]$

$t \rightarrow V[A; s, t]$ defines the total variation function of A .

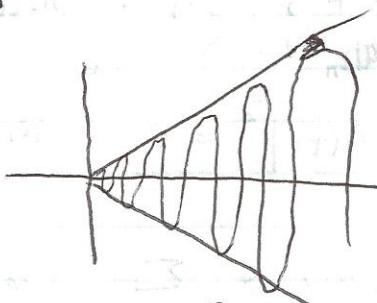
Say that " A has locally bounded variation" when $V[A; s, t] < \infty$ for all $t \in [s, \infty)$

Ex. 4.7.8 (a) $A: [0, \infty) \rightarrow \mathbb{R}$ is non-decreasing

$$V[A; s, t] = A(t) - A(s) < \infty$$

$\therefore A$ has L.B.V.

$$(b) A(t) = \int_0^t \frac{1}{t} dt = \ln t \quad t \in (0, \infty)$$



$$V[A; 0, t] = +\infty \text{ for all } t > 0$$

prop 4.7.9 Given $A, B: [0, \infty) \rightarrow \mathbb{R}$, $A \times B$ have Lbv. Then

(a) $A+B$ has Lbv.

$$V[A+B; s, t] \leq V[A; s, t] + V[B; s, t]$$

(b) for $c \in \mathbb{R}$ then $f_c A$ has Lbv.

$$V[cA] = |c| V[A; s, t]$$

(c) for $0 \leq s < t < u < \infty$ $V[A; s, u] = V[A; s, t] + V[A; t, u]$

Defn.

Given $A: [0, \infty) \rightarrow \mathbb{R}$, $0 \leq s < t < \infty$

Define $V[A; s, t] \triangleq \sup \left\{ \sum_{k=1}^n |A(t_k) - A(t_{k-1})| \right\}$ for $n \in \mathbb{N}$

(d) $V[A; s, t] = 0$ when $s = t$

$t \rightarrow V[A; 0, t]$ is non-decreasing.

Remark 4.7.11

put $A = A_1 - A_2$ where $A_1, A_2: [0, \infty) \rightarrow \mathbb{R}$ have LbV.
• A has LbV.

In particular, when A_1, A_2 are non-decreasing, then

A has LbV.

Lemma 4.7.12. Given $X \in \mathcal{M}_{\mathbb{C}}^{(1, f)}(P)$ st.

$|X(t, w)| \leq c < \infty$ for all (t, w) .

If $V[X(\cdot, w); 0, t] < \infty$ for all w , all $t \in [0, \infty)$

$P \{ w : X(t, w) = 0 \text{ for } t \in [0, \infty) \} = 1$

proof: Define $\{A_n(t) : t \in [0, \infty)\}$ as follows

$$A_n(t, w) \triangleq \sum_{0 \leq k < \infty} |X(t \wedge T_{k+1}^n, w) - X(t \wedge T_k^n, w)|^2$$

$$(T_k^n = k 2^{-n}, k, n = 0, 1, 2, \dots)$$

From Lemma 4.7.5, \exists a process $\{A(t)\}$ which is a Q.M.V. quadratic variation for X . st.

$$\lim_{n \rightarrow \infty} \mathbb{E} |A_n(t) - A(t)|^2 = 0 \text{ for each } t \in [0, \infty) \quad (1)$$

Moreover,

$$\begin{aligned} \text{But } A_n(t, w) &\leq \sum_{0 \leq k < \infty} \max_{0 \leq l < \infty} |X(t \wedge T_{k+l}^n - X(t \wedge T_k^n))|^2 \\ &\leq \max_{0 \leq l < \infty} |X(t \wedge T_{k+l}^n, w) - X(t \wedge T_k^n, w)|^2 \\ &\leq \max_{0 \leq l < \infty} |X(t \wedge T_{k+l}^n, w) - X(t \wedge T_l^n, w)| \sum_{0 \leq k < \infty} |X(t \wedge T_k^n, w) \\ &\quad - X(t \wedge T_l^n, w)| \end{aligned}$$



$$A_n(t, w) \leq \max_{0 \leq l \leq n} \{ |X(t + T_{l+1}^n, w) - X(t + T_l^n, w)| \}. V(X, w)$$

i.e. $\lim_{n \rightarrow \infty} A_n(t, w) = 0$ for all t, w \rightsquigarrow as $n \rightarrow \infty$ $\lim_{n \rightarrow \infty}$

From ① and ② (elementary meas. Theory)

$A(t) = 0$ a.s. for each t \dots ③

i.e. $A = 0$

Moreover from lemma 4.7.5, we know

$$X^2 - A \in M^C(\{\mathcal{F}_t\}, P)$$

$$\therefore X^2 \in M^C(\{\mathcal{F}_t\}, P)$$

$$\mathbb{E}[X^2(t)] = \mathbb{E}[X^2(0)] = 0 \text{ for all } t \in [0, \infty)$$

i.e. $X^2(t) = 0$ a.s. for all $t \in [0, \infty)$

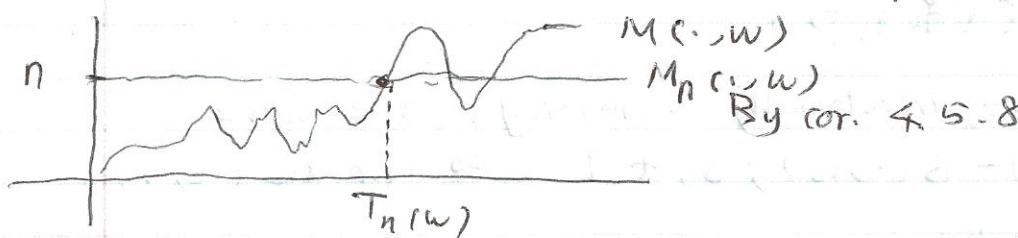
i.e. $X(t) = 0$ a.s. for all $t \in [0, \infty)$

Lemma 4.7.13. Given $M \in M_{loc}^{C, 0}(\{\mathcal{F}_t\}, P)$

such that $V[M(\cdot, w); 0, t] < \infty \quad t \in [0, \infty)$

$$P\{w : M(t, w) = 0 \quad t \in [0, \infty)\} = 1$$

Proof: Put $T_n(w) \triangleq \inf \{t \in [0, \infty) : |M(t, w)| \geq n\}$



Then T_n is a \mathcal{F}_t -stopping time

By prop. 4.6.9. we have that

$$M^{T_n} \in M^{C, 0}(\{\mathcal{F}_t\}, P)$$

$$M^{T_n}(t, w) \triangleq M(t + T_n(w), w)$$

Denote by $M_n(t, w)$

Moreover, the magnitude $|M_n(t, w)| \leq n$ all (t, w)

\subseteq by def. of V .

$V[\{M_n(\cdot, w), 0, t\}] \quad ? \quad V[\{M(\cdot, w), 0, t\}] < \infty$

I.e. from Lemma 4.7.12. by hypothesis

$P\{w : M_n(t, w) = 0 \text{ } t \in [0, \infty)\} = 1$

Observe that (by 4.6.9)

$\lim_{n \rightarrow \infty} T_n(w) = \infty$ From this, it easily follows that

$$\begin{aligned} & \{w : M(t, w) = 0 \text{ } t \in [0, \infty)\} \\ &= \bigcap_{n \geq 1} \{w : \underbrace{M_n(t, w) = 0}_{t \in [0, \infty)}\} \\ & \quad P(\cdot) = 1 \end{aligned}$$

$$\therefore P\{w : M(t, w) = 0 \text{ } t \in [0, \infty)\} = 1$$

Lemma 4.7.14. Let $x \in \mathcal{M}^c(\{\mathcal{F}_t\}, P)$ s.t. $|x(t, w)| \leq c < \infty$ for all (t, w)

suppose that $\{A(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$ are qu. processes for x .

Then A and B are indistinguishable.

Proof: We have that $x^2 - A \in \mathcal{M}^c(\{\mathcal{F}_t\}, P)$

$x^2 - B \in \mathcal{M}^c(\{\mathcal{F}_t\}, P)$

$A - B \in \mathcal{M}^c(\{\mathcal{F}_t\}, P)$

Since A & B are monotonically increasing, we have

$V[A(\cdot, w) - B(\cdot, w); 0, t] < \infty$ for all (t, w)

Result follows.

Thm: 4.7.15. (a) Let $x \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$

Then X has a quadratic variation process $\{A(t)\}$; furthermore.

If $\{B(t)\}$ is also a quadratic variation process for X, then A and B are indistinguishable.

"proof" (sketch) Define $T_n(w) \triangleq \inf \{t \in [0, \infty) \mid |x(t, w)| \geq n\}$

Then $0 \leq T_n \leq T_{n+1}$ and $T_n(w) \rightarrow +\infty$ for each w as $n \rightarrow \infty$

$$X_n(t, w) \triangleq X^{T_n}(t, w) = X(t \wedge T_n(w), w)$$

We have $X_n \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$ $|X_n(t, w)| \leq n$.

I.e. each X_n has a q.v. process $\{A_n(t)\}_{t \geq 0}$ which is unique to within indistinguishability.

$$(A_n)^T = (A_{n+1})^{T_n}$$

March 2, 2009. STAT 902 probability Theory II.

Defn. 4.7.1: Suppose $X \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$, then an R-valued process $\{A(t)\}$ is a quadratic variation process for X when

(a) $(A(t), \mathcal{F}_t)$ is adapted.

(b) $t \mapsto A(t, w)$ is continuous and non-decreasing for each w .

(c) $A(0) = 0$

(d) $X - A \in \mathcal{M}_{loc}^c$

Thm. 4.7.15 (a) suppose that $X \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$, then \exists a quadratic variation process for X which unique to within indistinguishability.

Remark 4.7.16.

$$\mathbb{E}[|x| < \infty] \text{ cf. } (\mathbb{E}[x | \mathcal{G}])$$

Given $X \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$, write

$\{\mathbb{E}[X](t), t \in [0, \infty)\}$ for a fixed but arbitrary choice of a quad.

var. process.

Recall prop. 4.1.8. Let $\{(w_t, \mathcal{F}_t)\}$ be a scalar wiener process, then

(a) $w \in \mathcal{M}_2^{c, 0}(\{\mathcal{F}_t\}, P)$

(b) for $X(t) = w(t) - t$, then $X \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$

put $A(t, w) \triangleq t$

i.e. $P\{w \mid \mathbb{E}[w](t) = t, t \geq 0\} = 1$
 $\mathbb{E}[w](t) = t$

Recall from Remark 4.6.3.

If $X \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$ and T is a \mathcal{F}_t -stopping time
then $X^T \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$

the process

Lemma 4.7.20. Under the preceding hypothesis, we have that.

$$\{[X^T](t), t \in [0, +\infty)\} \text{ and } \{[X](t \wedge T), t \in [0, \infty)\}$$

are indistinguishable.

Proof: we know that $\{(X^2(t) - [X](t)), \mathcal{F}_t\}$ is a continuous local martingale.

i.e. $\{(X^2(t \wedge T) - [X](t \wedge T)), \mathcal{F}_t\}$ is a continuous local MG. --- ①

Moreover, $\{((X^T(t))^2 - [X^T](t)), \mathcal{F}_t\}$ is a --- ②

Subtracting ① and ②

$$C(t) \triangleq [X^T](t) - [X](t \wedge T) \quad \text{--- ③}$$

from ① ② ③ $\{(C(t), \mathcal{F}_t)\}$ is a continuous local MG.

$$C(0) = 0$$

Moreover, $V[C(\cdot, w); 0, t] < \infty$ for all $t \geq 0$ all w .

$$\therefore C \equiv 0 \quad \square$$

Def. 4.7.21. suppose that X and y are $\in \mathbb{M}_{loc}^c(\{\mathcal{F}_t\}, P)$

then an \mathbb{R} -valued process $\{A(t)\}$ is co-quadratic variation process for X and y when

(a) $(A(t), \mathcal{F}(t))$ is adapted

(b) $t \mapsto A(t, w)$ is continuous and have locally bounded variation for each w

$$(c) A(0) = 0$$

$$(d) XY - A \in \mathbb{M}_{loc}^c(\{\mathcal{F}_t\}, P)$$

suppose that ξ and η are random variable on (Ω, \mathcal{F}, P) st.

$$\mathbb{E}[\xi^2] < \infty \quad \mathbb{E}[\eta^2] < \infty$$

$$\text{then } \text{Var}(\xi) \triangleq \mathbb{E}(\xi - E\xi)^2$$

$$\text{cov}(\xi, \eta) \triangleq \mathbb{E}[(\xi - E\xi)(\eta - E\eta)]$$

From the trivial identity

$$XY = \frac{1}{4} \{ (X+Y)^2 - (X-Y)^2 \} \quad X \text{ and } Y \in \mathbb{R}$$

We immediately see that

$$\text{cov}(\bar{X}, \bar{Y}) = \frac{1}{4} [\text{Var}(\bar{X} + \bar{Y}) - \text{Var}(\bar{X} - \bar{Y})]$$

Thm. 4.7.22. Given a $X, Y \in \mathcal{M}_{loc}^c(\mathbb{R}_+, P)$, define $\{A(t)\}$ by

$$A(t) \triangleq \frac{1}{4} \{[X+Y] - [X-Y]\}$$

Then $\{A(t)\}$ is a co-quadratic variation process for X and Y . Moreover, if $\{B(t)\}$ is another co-quadratic variation process, then $\{A(t)\}$ and $\{B(t)\}$ are indistinguishable.

Proof: Semi-trivial.

Remark 4.7.24: Given a pair of $X, Y \in \mathcal{M}_{loc}^c(\mathbb{R}_+, P)$, write $\{[X, Y](t), t \in [0, \infty)\}$ for a fixed but arbitrary choice of a co-quadratic variation process for X and Y .

Recall Prop. 4.1.10. Suppose that $\{(W_t, F_t)\}$ is a standard \mathbb{R}^d -valued BM/Wiener process and put

$$W_t = (W_t^1, W_t^2, \dots, W_t^d)$$

Then $\{W_j^i(t)W_k^k(t) - t\delta_{jk}, F_t\}$ is a continuous loc. martingale

$$[W^i, W^k](t) = t\delta_{jk}$$

Recall Lemma 4.7.5: Suppose that $X \in \mathcal{M}_{loc}^c(\mathbb{R}_+, P)$,

$$|X(t, w)| \leq C < \infty$$

Define processes $\{A_n(t), t \in [0, \infty)\}$, $n=1, 2, \dots$

by $A_n(t, w) = \sum_{0 \leq k \leq \infty} |X(t + \tau_{k+1}^n, w) - X(t + \tau_k^n, w)|^2$

$$\text{where } \tau_k^n = k2^{-n}, n, k=0, 1, 2, \dots$$

Then for each $t \in [0, \infty)$, the sequence of RV's $\{A_n(t), n=1, 2, \dots\}$ converges ($\text{in } L_2$) to $[X](t)$

$$\lim_{n \rightarrow \infty} \mathbb{E} \{ |A_n(t) - [X](t)|^2 \} = 0$$

Thm. 4.7.26. Suppose that $X \in \mathcal{M}_{loc}^c$ and define the sequence of processes $\{A_n(t), t \in [0, \infty)\}$ by (*)

Then for each $t \in [0, +\infty)$, the sequence of random variables $\{A_n(t), n=1, 2, \dots\}$ converges in probability to $[x](t)$.
 i.e. for each $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} P\{|A_n(t) - [x](t, \omega)| > \varepsilon\} = 0.$$

Thm 4.7.27. Suppose that X & Y are $\in \mathcal{M}_{loc}^e(\{\mathcal{F}_t\}, P)$, and define the sequence of processes $\{A_n(t), t \in [0, \infty)\}$ by

$$A_n(t, \omega) \triangleq \sum_{0 \leq k < n} [X(t \wedge T_{k+1}^n, \omega) - X(t \wedge T_k^n, \omega)] [Y(t \wedge T_{k+1}^n, \omega) - Y(t \wedge T_k^n, \omega)]$$

then

for each t , the sequence of RV's

$\{A_n(t), n=1, 2, \dots\}$ converges in probability to $[X \cdot Y](t)$.

Lemma 4.7.28: Take $X, Y, Z \in \mathcal{M}_{loc}^e$, Then

(a) $\{[x](t)\} \& \{[x, x](t)\}$ are indistinguishable

$$[x] = [x, x]$$

(b) $\{[x, y](t)\} \& \{[y, x](t)\}$ are indistinguishable.

(c) Take $\alpha \& \beta \in \mathbb{R}$, Then the processes

$$\{[\alpha X + \beta Y, Z](t)\}$$

$\{\alpha[x, z]_t + \beta[y, z](t)\}$ are indistinguishable

(d) When $0 \leq s < t < \infty$

$$|[x, y](t) - [x, y](s)| \leq \{[x, y](t) - [x, y](s)\}^{1/2} \{[y](t) - [y](s)\}$$

(e) If $y(t) = Y_0$ for all $t \in [0, \infty)$, then

$$[x, y] = 0$$

Proof of part (d): Fix $0 \leq s < t < \infty$, $\lambda \in \mathbb{R}$

$$0 \leq [x + \lambda Y](t) - [x + \lambda Y](s) \quad \text{as. --- ①}$$

$$\text{From (a)} \quad [x + \lambda Y](t) = [x + \lambda Y, x + \lambda Y](t)$$

$$\text{from (c)} \quad = [x, x](t) + 2\lambda[x, Y](t) + \lambda^2[Y, Y](t)$$

$$= [x](t) + 2\lambda[x, Y](t) + \lambda^2[Y](t) \quad \text{--- ②}$$

as.

Similarly, $[x+\lambda y](s) = [x](s) + 2\lambda[x,y](s) + \lambda^2[y](s)$... ③
 put ② and ③ in ① we get

$$0 \leq [x](t) - [x](s) + 2\lambda([x,y](t) - [x,y](s)) + \lambda^2[y](t) - [y](s)$$

for all $\lambda \in \mathbb{R}$

Result follows from the discriminant of quadratic function of λ
 on the RHS of ④

Recall lemma 4.7.20.

Let $x \in M_{loc}^c(\{\mathcal{F}_t\}, P)$, and let T be a stopping time
 then $[x^T] = [x](t \wedge T)$ indistinguishable.

Lemma 4.7.32:

Fix x and $y \in M_{loc}^c$ and a stopping time T .

$$\{[x, y](t \wedge T), t \in [0, \infty)\}$$

$\{[x, y^T](t), t \in [0, \infty)\}$ are all indistinguishable.

$$\{[x^T, y](t), t \in [0, \infty)\}$$

$$\{[x^T, y^T](t), t \in [0, \infty)\}$$

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Ch 5 Stochastic Integration

1. Lebesgue-style integration.

Recall for $x \in \mathbb{R}$. $[x]^+ \stackrel{\Delta}{=} \max(x, 0)$

$$[x]^-=\max(-x, 0)$$

Defn. 5.1.1 + Remark 5.1.2: Given $A: [0, \infty] \rightarrow \mathbb{R}$,

$$\bar{A}(t) \stackrel{\Delta}{=} V[A; 0, t] = \sup_n \sum_{k=1}^n |A(t_k) - A(t_{k-1})|$$

total variation of A 

$$\text{Define } A_+(t) \stackrel{\Delta}{=} \sup_n \sum_{k=1}^n [A(t_k) - A(t_{k-1})]^+$$

Lemma 5.1.3: Given $A: [0, \infty] \rightarrow \mathbb{R}$

$$\text{Then } \bar{A}(t) = A_+(t) + A_-(t) \quad |x| \leq [x]^+ + [x]^-$$

Moreover, if A has l.b.v. then

$$x = [x]^+ - [x]^-$$

$$A(t) = A(0) + A_+(t) - A_-(t)$$

Remark: since A has l.b.v. we have that

$$\text{i.e. } A(t) < \infty \quad t \in [0, +\infty) \quad \text{and} \quad A_+(t) + A_-(t) < \infty \quad t \in [0, +\infty)$$

$$\text{i.e. } A_+(t) < \infty, \quad A_-(t) < \infty \quad t \in [0, +\infty)$$

Lemma 5.1.6 = suppose $A: [0, \infty) \rightarrow \mathbb{R}$ is continuous with l.b.v.

then $A \rightarrow A_+, A_-$ are continuous non-decreasing

Moreover,

$$A(t) = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} |A(t \wedge T_k^n) - A(t \wedge T_k^{n-1})|$$



$$A_+(t) = \lim_{n \rightarrow \infty} \sum_{0 \leq k < \infty} [A(t \wedge T_k^n) - A(t \wedge T_k^{n-1})]^+$$

Thm 5.1.7 suppose $A: [0, \infty) \rightarrow \mathbb{R}$ is continuous and non-decreasing.

then there exists a unique measure $\mu: (\mathcal{B}[0, \infty), \mathcal{A}) \rightarrow [0, \infty)$ st.

$$\int_0^t \mu([s, +\infty)) dA(s) = A(t) - A(s) \quad \mu([t, +\infty)) = 0$$

call μ the Lebesgue-Stieltjes measure of A , denote by M_A

$A(t) \triangleq t \rightarrow$ Lebesgue measure.

L-S Integrals:

Step 1: Given continuous non-decreasing $A: [0, \infty) \rightarrow \mathbb{R}$

$\Phi: \mathcal{B}[0, \infty) \rightarrow [0, \infty]$ is $\mathcal{B}[0, \infty)$ -measurable

Then the integral $\int_{[0, \infty)} \Phi dM_A$ exists in $[0, \infty]$

We call this integral the L-S integral of Φ with respect to A

and denote by $\int_0^\infty \Phi(s) dA(s) = \int_{[0, \infty)} \Phi dM_A$

Step 2: Given $A: [0, \infty) \rightarrow \mathbb{R}$ is continuous (non-decreasing)

$\Phi: [0, \infty) \rightarrow \overline{\mathbb{R}}$ is $\mathcal{B}[0, \infty)$ -meas.

the L-S integral of Φ with respect to A is defined by

$$\int_0^\infty \Phi(s) dA(s) \triangleq \int_0^\infty \Phi_+(s) dA(s) - \int_0^\infty \Phi_-(s) dA(s)$$

provided that $\int_0^\infty \Phi_+(s) dA(s) < \infty$

equivalently, $\int_0^\infty |\Phi(s)| dA(s) < \infty$

Step 3: Given $A: [0, \infty) \rightarrow \mathbb{R}$ is continuous with (b.v.)

$\Phi: [0, \infty) \rightarrow \mathbb{R}$ is $\mathcal{B}([0, \infty))$ -meas.

The L-S integral of Φ wrt. A is defined by

$$\int_0^\infty \Phi(s) dA(s) = \int_0^\infty \Phi(s) dA_+(s) - \int_0^\infty \Phi(s) dA_-(s)$$

provided each term on the right exists and finite.

Recall that $\int_0^\infty \Phi(s) dA \pm(s)$ is the Lebesgue integral

$\int_{[0, \infty)} \Phi d\mu_{A\pm}$ which exists provided that

$$\int_{[0, \infty)} |\Phi| d\mu_{A\pm} < \infty$$

It is semi-trivial to show that

$$\int_0^\infty |\Phi| d\mu_{A\pm} < \infty \text{ iff } \int_0^\infty |\Phi(s)| dA(s) < \infty$$

To summarize, we define

$$\int_0^\infty \Phi(s) dA(s) = \int_0^\infty \Phi(s) dA_+(s) - \int_0^\infty \Phi(s) dA_-(s)$$

provided $\int_0^\infty |\Phi(s)| dA(s) < \infty$

$$\text{i.e. } \int_0^\infty \Phi(s) dA(s) = \int_0^\infty \Phi_+(s) dA_+(s) - \int_0^\infty \Phi_-(s) dA_-(s)$$

$$- [\int_0^\infty \Phi_+(s) dA_-(s) - \int_0^\infty \Phi_-(s) dA_-(s)] \quad (*)$$

It is semi-trivial to show that

$\int_0^\infty |\Phi(s)| dA(s)$ is equivalent to each of the four integrals

on the right hand side of $(*)$ being finite.

Lemma 5.1.12: Given $A: [0, \infty) \rightarrow \mathbb{R}$ continuous with (b.v.)

$\Phi: [0, \infty) \rightarrow \mathbb{R}$ is $\mathcal{B}([0, \infty))$ -meas. Such that

$$\int_0^t |\Phi(s)| dA(s) < \infty \text{ for all } t \in [0, \infty), \text{ then}$$

$\int_0^t \Phi(s) dA(s)$ exists

Given $A: [0, \infty) \rightarrow \mathbb{R}$ continuous with (b.v.) in \mathbb{R} for each $t \in [0, \infty)$

$\Phi: [0, \infty) \rightarrow \mathbb{R}$ $\mathcal{B}([0, \infty))$ -meas.

We define $\int_0^t \Phi(s) dA(s) \triangleq \int_0^\infty \Phi(s) \mathbb{I}_{\{s \leq t\}} dA(s)$

provided that $\int_0^\infty |\Phi(s) \mathbb{I}_{\{s \leq t\}}| d\tilde{A}(s)$ is finite.

i.e. provided $\int_0^t |\Phi(s)| d\tilde{A}(s) < \infty$

$$\int_0^t |\Phi| d\tilde{A} < \infty$$

Moreover, the mapping $B: [0, \infty) \rightarrow \mathbb{R}$ is continuous with Lbv.

and $\tilde{B}(t) = \int_0^t |\Phi(s)| d\tilde{A}(s)$

likewise, there formulae for $B(t)$

Thm 5.3.13. (b) suppose $A: [0, \infty) \rightarrow \mathbb{R}$ is a continuous with Lbv.

$\Phi: [0, \infty) \rightarrow \overline{\mathbb{R}}$ is $B[0, \infty)$ -meas. st.

$$\int_0^t |\Phi(s)| dA(s) < \infty \text{ for all } t \in [0, \infty)$$

put $B(t) \triangleq \int_0^t \Phi(s) dA(s) \quad t \in [0, \infty)$

suppose in addition, that

$\Psi: [0, \infty) \rightarrow \overline{\mathbb{R}}$ is $B[0, \infty)$ -meas. st.

$$\int_0^t |\Psi(s)| dB(s) < \infty \quad t \in [0, \infty)$$

put $C(t) \triangleq \int_0^t |\Psi(s)| dB(s) \quad t \in [0, \infty)$

Then C is \mathbb{R} -valued continuous with Lbv. moreover,

$$\int_0^t \Psi(s) dB(s) = \int_0^t \Psi(s) \underline{\Phi(s) dA(s)}$$

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Thm 5.1.15. Given $A, B: [0, \infty) \rightarrow \mathbb{R}$ which are continuous with Lbv.

Then $A(t)B(t) = A(0)B(0) + \int_0^t A(s) dB(s) + \int_0^t B(s) dA(s)$

Preview: Given $X, Y \in M_{bc}^+(\mathcal{F}_T), P$

i.e. $X(\cdot, w), Y(\cdot, w)$ are continuous.

can we write

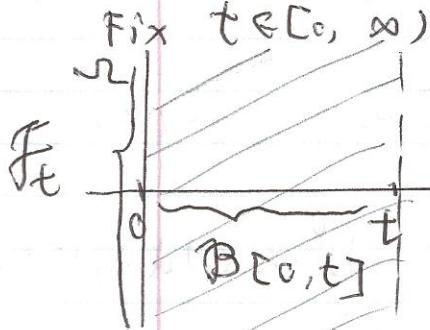
$$X(t, w)Y(t, w) = X(0, w)Y(0, w) + \int_0^t X(s, w) dY(s, w) + \int_0^t Y(s, w) dX(s, w) + [X, Y](t)$$

"Ito integration by parts formula,"

$$\int_0^t W(s, w) dW(s, w)$$

§ 5.2 Pathwise L-S integration.

Recall Defn. 3.1.21 Given a filtration $\{\mathcal{F}_t\}$ in $(\Omega, \mathcal{F}, \mathbb{P})$, a process $(\Phi(t), \mathcal{F}_t)$ is progressively measurable when



Then $(\Phi(t), \mathcal{F}_t)$ is progressively measurable means the following:

$$(s, w) \mapsto \Phi(s, w) : [0, t] \otimes \Omega \rightarrow \mathbb{R}$$

is $B[0, t] \otimes \mathcal{F}_t$ - meas.

Then $w \mapsto \int_0^t \Phi(s, w) ds$ measure does not change

is \mathcal{F}_t - meas. (Tonelli)

Thm 5.2.4: (true-charged Tonelli)

suppose that $(A(t), \mathcal{F}_t)$ and $(\Phi(t), \mathcal{F}_t)$ are prog-meas. such that

(a) $t \mapsto A(t, w)$ is continuous and non-decreasing for each w .

(b) $A(t, w) \in [0, \infty] \quad \forall (t, w) \in [0, \infty) \otimes \Omega$

Define $\int_0^t \Phi(s, w) dA(s, w)$ exists in $[0, \infty]$

Then $w \mapsto \int_0^t \Phi(s, w) dA(s, w)$ is \mathcal{F}_t - meas.

Remark 5.2.5 Given a filtration $\{\mathcal{F}_t\}$ in $(\Omega, \mathcal{F}, \mathbb{P})$. write

$\underset{\sim}{FV}^C(\{\mathcal{F}_t\})$ to denote the set of all processes $\{A(t)\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ st.

(a) $(A(t), \mathcal{F}_t)$ is \mathcal{F}_t - adapted

(b) $t \mapsto A(t, w)$ is continuous with l.b.v. for each w

$$A \in \underset{\sim}{FV}^C(\{\mathcal{F}_t\})$$

Lemma 5.2.6 Take $A \in \mathbb{F}_\sim^C(\{\mathcal{F}_t\})$, then
 $\check{A}(\cdot, w)$, $A_+(\cdot, w)$ & $A_-(\cdot, w)$ are continuous non-decreasing
 Moreover, $(\check{A}(\cdot, w), \mathcal{F}_t)$, $(A_+(\cdot, w), \mathcal{F}_t)$ are adapted processes.

Recall from § 5.1:

Given a mapping $A: [0, \infty) \rightarrow \mathbb{R}$ is continuous with LBV.
 $\Phi: [0, \infty) \rightarrow \mathbb{R}$ is Borel $[0, \infty)$ -meas.

st. $\int_0^t |\Phi(s)| dA(s) < \infty \text{ for all } t \in [0, \infty) \quad \dots (*)$

$$\text{Then } B(t) \triangleq \int_0^t \Phi(s) dA(s) = \left\{ \int_0^t \Phi_+(s) d\check{A}_+(s) - \int_0^t \Phi_-(s) d\check{A}_-(s) \right\} \\ - \left\{ \int_0^t \Phi_+(s) dA(s) - \int_0^t \Phi_-(s) d\check{A}(s) \right\}$$

From (*) we see that each integral on RHS takes values in $[0, \infty)$, i.e., the RHS is well-defined.

Then for $B(t) \triangleq \int_0^t \Phi(s) dA(s)$ the function B is continuous with LBV.

Remark 5.2.7 suppose $A \in \mathbb{F}_\sim^C(\{\mathcal{F}_t\})$, $(\Phi(t), \mathcal{F}_t)$ is
 Prog. meas. st.

$\int_0^t |\Phi(s, w)| d\check{A}(s, w) < \infty \text{ for all } t \in [0, \infty) \quad (**)$

we and all $w \in \mathbb{R}$
 we define pathwise integral

\mathcal{F}_t -meas.

$$\int_0^t \Phi(s, w) dA(s, w) \triangleq \left\{ \int_0^t \Phi_+(s, w) d\check{A}_+(s, w) - \int_0^t \Phi_-(s, w) d\check{A}_-(s, w) \right\} \\ - \left\{ \int_0^t \Phi_+(s, w) dA(s, w) - \int_0^t \Phi_-(s, w) d\check{A}_-(s, w) \right\}$$

thus $B(t, w) \triangleq \int_0^t \Phi(s, w) dA(s, w)$

we have $B \in \mathbb{F}_\sim^C(\{\mathcal{F}_t\})$

Typically, in place of (**), we have:

for each $t \in [0, \infty)$

$$\mathbb{P}\{w : \int_0^t |\Phi(s, w)| d\check{A}(s, w) < \infty\} = 1$$

$$\text{I.e. } N_t \stackrel{\Delta}{=} \{w : \int_0^t |\Phi(s, w)| dA(s, w) = 0\}$$

then $P(N_t) = 0$ for each $t \in [0, \infty)$

Observe that $0 \leq t_1 < t_2 < \infty \Rightarrow N_{t_1} \subset N_{t_2}$

put

$$N \stackrel{\Delta}{=} \bigcup_{0 \leq n < \infty} N_n \quad P(N_n) = 0$$

\uparrow integers

$$\therefore P(N) = 0$$

Moreover, $N_t \subseteq N$ for all $t \in [0, \infty)$

For $w \notin N$, we have $w \notin N_t$ for all $t \in [0, \infty)$

$$\text{i.e. } \int_0^t |\Phi(s, w)| dA(s, w) < \infty \text{ for all } t \in [0, \infty)$$

For $w \in N$, PUT / Define $\int_0^t \Phi(s, w) dA(s, w) \stackrel{\Delta}{=} 0$ for all $t \in [0, \infty)$

i.e. the pathwise L-S integral

$\int_0^t \Phi(s, w) dA(s, w)$ is well-defined for all $t \in [0, \infty)$

when $w \notin N$.

Then, the mapping $t \mapsto B(t, w) \stackrel{\Delta}{=} \int_0^t \Phi(s, w) dA(s, w)$

is continuous with Lbv. for each w .

FROM NOW ON, WE SUPPOSE THAT

$$\{A \in \mathcal{F} \mid P(A) = 0\} \subset \mathcal{F}_0$$

$$N \cap \mathcal{F}_0 \subset \mathcal{F}_t$$

Then it is an easy exercise in measure theory to show that

$w \mapsto \underbrace{\int_0^t \Phi(s, w) dA(s, w)}_{B(t, w)}$ is \mathcal{F}_t -meas.

i.e. $B \in \mathbb{F}_V^c(\{\mathcal{F}_t\})$ \mathcal{F}_t -adapted
finite variation

§5.3 Stochastic Integration:

Given a progr. meas. process

$(\Phi(t), \mathcal{F}_t)$ and $X \in M_{loc}^c(\{\mathcal{F}_t\}, P)$

How do we define a stochastic integral

$$\int_0^t \Phi(s, w) dX(s, w)$$

Kunita-Watanabe theory of stochastic integration
(More Modern Approach)

§5.3. Recall Given a process $A \in FV^c(\{\mathcal{F}_t\}, P)$

a progr. meas. $(\Phi(t), \mathcal{F}_t)$ st.

$$\int_0^t |\Phi(s)| dA(s) < \infty \text{ as. for each } t \in [0, \infty)$$

Then we can define the Lebesgue-Stieltjes integral

$$\int_0^t \Phi(s) dA(s) \text{ for individual}$$

Suppose given a process $X \in M_{loc}^c(\{\mathcal{F}_t\}, P)$

can we define $\int_0^t \Phi(s) dX(s)$ for individual w ?

Unless X is trivial, we have

$$X(\cdot, w) = +\infty \quad X_{\pm}(\cdot, w) = +\infty$$

i.e. the L-S measures $\mu_{X_{\pm}}(\cdot, w)$ does not exist

Follow Ito-Kunita-Watanabe

Condition 5.3.1 The filtration $\{\mathcal{F}_t\}$ in (Ω, \mathcal{F}, P) is st. \mathcal{F}_0 includes all P -null events of \mathcal{F} .

Fix $x \in M_{loc}^c(\{\mathcal{F}_t\}, P)$

$$\sup_t \mathbb{E}[|x(t)|^2] < \infty$$

Lévy Ito

$$\int_0^t \Phi(s, w) dW(s, w)$$

Bm.

Defn. 5.3.8. Given $X \in M_{loc}^{c, 0}(\{\mathcal{F}_t\}, P)$

$L^2(X, \{\mathcal{F}_t\}, P)$ (or $L^2(x)$ for short)

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Ch5 Stochastic Integration

Denotes the set of all processes $\{\hat{X}(t)\}$ on (Ω, \mathcal{F}, P) such that

- (i) $(\hat{X}(t), \mathcal{F}_t)$ is progr. meas.
- (ii) $E\left[\int_0^\infty |\hat{X}(s)|^2 d[X](s)\right] < \infty$

Recall Thm 1.4.13:

Given an integrable RV. X on (Ω, \mathcal{F}, P) and a σ -alg. $\mathcal{G} \subset \mathcal{F}$. Then \exists some integrable \mathcal{G} -meas. RV. Z such that

$$E[Z; A] = E[X; A] \text{ for all } A \in \mathcal{G}.$$

Thm 5.3.14 (Kunita-Watanabe Theorem)

Given $X \in M_{2b}^{c,0}$ and $\hat{X} \in L^2(X)$ then \exists some $M \in M_{2b}^{c,0}$ with the following properties:

for each and every $Z \in M_{2b}^c$,

$$[M, Z] = \int \hat{X}(s) d[X, Z](s) \quad \text{--- (1)}$$

Furthermore, if $\tilde{M} \in M_{2b}^{c,0}$, st.

$$[\tilde{M}, Z] = \int \hat{X}(s) [X, Z](s) \text{ for each } Z \in M_{2b}^c, \text{ then } M \text{ and } \tilde{M} \text{ are indistinguishable.}$$

Proof of Uniqueness

From (1) & (2), we have $[M, Z] = [\tilde{M}, Z]$ for each $Z \in M_{2b}^c$

$$\text{i.e. } [M - \tilde{M}, Z] = 0 \text{ for all } Z \in M_{2b}^c.$$

$$\text{Take } Z = M - \tilde{M}$$

$$\text{i.e. } \underbrace{[M - \tilde{M}, M - \tilde{M}]}_{[M - \tilde{M}]} = 0$$

$$\text{But } (M - \tilde{M})^2 = [M - \tilde{M}] \in M_{2b}^{c,0}$$

$$\text{Therefore } (M - \tilde{M})^2 \in M_{2b}^{c,0} \quad E[(M(t) - \tilde{M}(t))^2] = 0$$

$$\mathbb{E}[|z|^2] = 0$$

i.e. $M(t) = \tilde{M}(t)$ as, for each t

i.e. \tilde{M} is a modification of M .

since path of M & \tilde{M} are continuous, see from prop. 3.1.1

that M & \tilde{M} are indistinguishable. \square

Remark

Definition: 3.4.15. Given $X \in M_{2b}^{C,0}$, $\Phi \in L^2(x)$

any $M \in M_{2b}^{C,0}$ given by theorem 3.3.14 is called an Ito^2 stochastic

integral of Φ with respect to X . we will denote this by

$$\{(\Phi \circ X)(t) \text{ for } t \int_0^t \Phi(s) dX(s)\}$$

Remark (*): Given $X \in M_{2b}^{C,0}$, $\Phi \in L^2(x)$, there exists some element

$\Phi \circ X \in M_{2b}^{C,0}$ with the property that

$$[\Phi \circ X, z] = \int \Phi(s) d[X, z] \quad \text{for each } z \in M_{2b}^C$$

This element is unique to within indistinguishability.

Thm 5.3.17:

Given $X \in M_{2b}^{C,0}$, $\Phi \in L^2(x)$ ($\because \Phi \circ X \in M_{2b}^{C,0}$)

$Y \in M_{2b}^{C,0}$, $\Psi \in L^2(Y)$ ($\because \Psi \circ Y \in M_{2b}^{C,0}$)

$$\text{Then } [\Phi \circ X, \Psi \circ Y] = \int \Phi(s) \Psi(s) d[X, Y](s)$$

Proof: take $Z \stackrel{\Delta}{=} \Psi \circ Y$ in Remark (*):

$$[\Phi \circ X, \Psi \circ Y] = \int \Phi(s) d[X, \Psi \circ Y] \dots \text{--- } \textcircled{1}$$

Moreover, $[X, \Psi \circ Y] = [\Psi \circ Y, X]$

by Remark
(*) again $= \int \Psi(s) d[X, Y] \dots \text{--- } \textcircled{2}$

Result follows from $\textcircled{1}$ $\textcircled{2}$ and chain rule for L^2 -integrals \square

Corollary 5.3.18:

Given $x \in M_{2b}^{c,0}$ $\Phi \in L^2(X)$

$$[\Phi \circ x] = [\Phi \cdot x, \Phi \cdot x] = \int_0^1 |\Phi(s)|^2 d[x](s)$$

Ito Isometry.

Theorem 5.3.20: Given

$$x \in M_{2b}^{c,0}, \Phi \in L^2(X) \quad (\therefore \Phi \circ x \in M_{2b}^{c,0})$$

$$\psi \in L^2(\Phi \circ x) \quad (\therefore \psi \circ (\Phi \circ x) \in M_{2b}^{c,0})$$

Then $\psi \circ (\Phi \circ x) = (\psi \Phi) \circ x$

i.e. for $y = \Phi \circ x = \int_0^1 \Phi(s) dx(s)$

then $\int_0^1 \psi(s) dy(s) = \int_0^1 \psi(s) \Phi(s) dx(s)$

Proof: From Remark (*)

$$[\psi \circ (\Phi \circ x), z] = \int_0^1 \psi(s) d[\Phi \circ x, z](s) \quad \text{--- (1)}$$

for each $z \in M_{2b}^c$

From Remark (*) again

$$[\Phi \circ x, z] = \int_0^1 \Phi(s) d[x, z](s) \quad \text{for each } z \in M_{2b}^c \quad \text{--- (2)}$$

From (1) (2) and chain rule for L-S integrals.

$$[\psi \circ (\Phi \circ x), z] = \int_0^1 \psi(s) \Phi(s) d[x, z](s) \quad \text{--- (3)}$$

for each $z \in M_{2b}^c$

Moreover, From Remark (*) again -

$$[(\psi \Phi) \circ x, z] = \int_0^1 \psi(s) \Phi(s) d[x, z] \quad \text{--- (4)}$$

for all $z \in M_{2b}^c$

From (3), (4) and Uniqueness part of the theorem 5.3.14

We get

$$\psi \circ (\Phi \circ x) = (\psi \Phi) \circ x \quad \text{--- (5)}$$

Theorem 5.3.21: Given $x \in M_{2b}^{c,0}$, Φ and $\psi \in L^2(X)$

then for α and $\beta \in \mathbb{R}$

$$(\alpha \Phi + \beta \Psi) \circ X = \alpha (\Phi \circ X) + \beta (\Psi \circ X)$$

Thm. 5.3.13 Given two continuous $X \in \mathcal{M}_{2b}^{C,0}$
 $\Phi \in L^2(X) \cap L^2(Y)$

(Therefore $\Phi \circ X \circ Y \in \mathcal{M}_{2b}^{C,0}$)

$$\Phi \circ (\alpha X + \beta Y) = \alpha (\Phi \circ X) + \beta (\Phi \circ Y)$$

$$\left\{ \begin{array}{l} X \in \mathcal{M}_{2b}^{C,0} \\ \Phi \in L^2(X) \end{array} \right\} \subseteq \mathcal{M}_{loc}^C \cdot L_{loc}^2(X)$$

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Given $X \in \mathcal{M}_{2b}^{C,0}$ $\Phi \in L_2(X)$

$$\text{i.e. } E\left[\int_0^\infty |\Phi(s)|^2 d[X]\right] < \infty$$

Thm. 5.3.14 \exists some $M \in \mathcal{M}_{2b}^{C,0}$ st. for each $Z \in \mathcal{M}_{2b}^{C,0}$

We have

$$[M, Z] = \int_0^\infty \Phi(s) d[X, Z]$$

Furthermore, M is unique to within indistinguishability.

Remark 5.3.23.

Given $X \in \mathcal{M}_{2b}^{C,0}(\{\mathcal{F}_t\}, P)$, $\Phi \in L^2(X(\mathcal{F}_+), P)$ T is a \mathcal{F}_t -stopping time.

The truncation of Φ at T is defined by

$$\Phi_{[0,T]}(t, \omega) \triangleq \begin{cases} \Phi(t, \omega) & t \leq T(\omega) \\ 0 & t > T(\omega) \end{cases}$$

clear that $\Phi_{[0,T]} \in L^2(X)$

$$\therefore \Phi_{[0,T]} \circ X \in \mathcal{M}_{2b}^{C,0}$$

Moreover, it is also clear $\Phi \in L^2(X^T)$

$$\therefore \Phi \circ (X^T) \in \mathcal{M}_{2b}^{C,0}$$

Finally, $(\Phi \circ X)^T \in \mathcal{M}_{2b}^{C,0}$

Thm 5.3.24: $\Phi_{[0,T]} \circ X$, $\Phi \circ (X^T)$, $(\Phi \circ X)^T$ are all indistinguishable

Defn. 5.3.27: Given $X \in \mathcal{M}_{loc}^{C,0}$, denote by $L_{loc}^2(X, \{\mathcal{F}_t\}, P)$

(or $L_{loc}^2(X)$)

be set of all processes $\{\Phi(t)\}$ st.

(a) $(\Phi(t), \mathcal{F}_t)$ - progressively measurable

(b) $\int_0^t |\Phi(s)|^2 d[X](s) < \infty$ as, for each $t \in [0, \infty)$

$$E \left\{ \int_0^\infty |\Phi(s)|^2 d[X](s) \right\} < \infty$$

Thm. 5.3.31. (ktw) : Given a continuous local martingale $X \in \mathcal{L}^2$ note

$$M_{loc}^{c,0} \ni \Phi \in L_{loc}^2(X)$$

\exists some $M \in M_{loc}^{c,0}$ st. for each $\varphi \in M_{loc}^{c,\infty}$ we have

$$[M, \varphi] = \int_0^{\cdot} \Phi(s) d[X, \varphi]$$

Moreover, M is unique to within indistinguishability.

Defn. 5.3.32. Given $X \in M_{loc}^{c,0}$, $\Phi \in L_{loc}^2(X)$.

we denote by $\Phi \circ X$ or $\int \Phi(s) dX(s)$ an arbitrary choice of M given by Thm 5.3.31.

Thm 5.3.33 - 5.3.38

(a) $X \in M_{loc}^{c,0}$, $\Phi \in L_{loc}^2(X)$, T is a stopping time

Then $(\Phi[0, T])^0 X \Phi^0(X^T)$, $(\Phi \circ X)^T$ are indistinguishable.

(b) $X \in M_{loc}^{c,0}$, $\Phi \in L_{loc}^2(X)$. ($\because \Phi \circ X \in M_{loc}^{c,0}$)

$Y \in M_{loc}^c$, $\psi \in L_{loc}^2(Y)$ ($\because \forall \varphi \in M_{loc}^{c,c}$)

$$[\Phi \circ X, \psi \circ Y] = \int \Phi(s) \psi(s) d[X, Y](s)$$

(c) $X \in M_{loc}^{c,0}$, $\Phi \in L_{loc}^2(X)$ ($\because \Phi \circ X \in M_{loc}^{c,0}$)

$\psi \in L_{loc}^2(\Phi \circ X)$ ($\because \psi \circ (\Phi \circ X) \in M_{loc}^{c,0}$)

Then $\psi \circ (\Phi \circ X) = (\psi \Phi) \circ X$

i.e. $\int_0^{\cdot} \psi(s) d(\Phi \circ X)(s) = \int_0^{\cdot} \psi(s) \Phi(s) dX(s)$ Chain Rule

(d) $X \in M_{loc}^{c,0}$, $\Phi, \psi \in L_{loc}^2(X)$.

Then for $\alpha, \beta \in \mathbb{R}$

$$(\alpha \Phi + \beta \psi) \circ X = \alpha (\Phi \circ X) + \beta (\psi \circ X)$$

(e) $X \otimes Y \in M_{loc}^{c,0}$, $\Phi \in L^2_{loc}(x) \cap L^2_{loc}(y)$

Then for $\alpha, \beta \in \mathbb{R}$

$$\Phi \circ (\alpha X + \beta Y) = \alpha(\Phi \circ X) + \beta(\Phi \circ Y)$$

Recall: (i) when X is $\alpha \in M_{2b}^{c,0}$, & $\Phi \in L^2(x)$, then

$$\Phi \circ X \in M_{2b}^{c,0}$$

(ii) when $X \in M_{loc}^{c,0}$ & $\Phi \in L^2_{loc}(x)$, then $\Phi \circ X \in M_{loc}^{c,0}$

Question: Given $X \in M_{loc}^{c,0}$, what additional conditions must be imposed on Φ to ensure that $\Phi \circ X \in M_2^{c,0}$ boundedness

Corollary 5.3.41: Given $X \in M_{loc}^{c,0}$, suppose that note!

(a) $(\Phi(t), \mathcal{F}_t)$ is pro. meas.

(b) $E \left\{ \int_0^t |\Phi(s)|^2 d[X](s) \right\} < \infty$ for all $t \in [0, \infty)$

Then $\Phi \circ X \in M_2^{c,0}$ — generic mg.

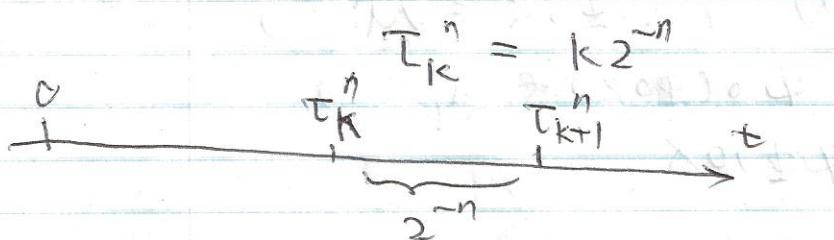
i.e. what can we say about $E((\Phi \circ X)(t)) = 0$ for all $t \in [0, \infty)$

Theorem 5.3.50: Suppose $X \in M_{loc}^{c,0}$, $\Phi \in L^2_{loc}(x)$.

$t \rightarrow \Phi(t, w)$ are continuous. can be omitted, but the proof becomes much harder.

for each $n = 1, 2, \dots$

$$M_n(t, w) \triangleq \sum_{0 \leq k \leq n} \Phi(t + 1T_k^n, w) [X(t + 1T_k^n, w) - X(t, w)]$$



For each $t \in [0, \infty)$ the sequence of rvs $\{M_n(t)\}_{n=1,2,\dots}$ converges in probability to $\Phi \circ X(t)$

Rewz + Far --- a good reference, deep.

§5.4 Ito's formula

73.

Defn. 5.4.1 Suppose condition 5.3.1 on $\{\mathcal{F}_t\}$, an $\{\mathcal{F}_t\}$ -semi-martingale is a continuous process $\{X(t)\}$ of the form

$$X(t) = X_0 + M(t) + A(t)$$

where (i) X_0 is a \mathcal{F}_0 -meas. RV.

(ii) $M \in \tilde{M}_{loc}^{c,0}$

(iii) $A \in \overset{\sim}{EV}^{c,0}$

Note that, $X(0) = \bar{X}_0$

Remark 5.4.2.

Given a filtration $\{\mathcal{F}_t\}$ in (Ω, \mathcal{F}, P) subject to condition 5.3.1

we write $\overset{\sim}{SM}^c(\{\mathcal{F}_t\}, P)$ to denote the set of all continuous $\{\mathcal{F}_t\}$ -semimartingales.

Remark 5.4.3. Given $X \in \overset{\sim}{SM}^c(\{\mathcal{F}_t\}, P)$, suppose

$$X(t) = X_0 + M(t) + A(t) \quad \dots \textcircled{1}$$

$$X(t) = \bar{X}_0 + \bar{M}(t) + \bar{A}(t) \quad \dots \textcircled{2}$$

Then $X_0 = \bar{X}_0$ $M(t) = \bar{M}(t)$ $\bar{A} = A$ (indistinguishability)

Observe that

$$X(0) = X_0 = \bar{X}_0 \quad \dots \textcircled{3}$$

From $\textcircled{1}, \textcircled{2}, \textcircled{3}$

$$M(t) + A(t) = \bar{M}(t) + \bar{A}(t)$$

$$\therefore M(t) - \bar{M}(t) = \bar{A}(t) - A(t)$$

$$\underbrace{M}_{\in \tilde{M}_{loc}^{c,0}} - \underbrace{\bar{M}}_{\in \overset{\sim}{EV}^{c,0}} = \bar{A} - A$$

$$M - \bar{M} \in \tilde{M}_{loc}^{c,0} \cap \overset{\sim}{EV}^{c,0}$$

$$M - \bar{M} = 0 \quad \text{i.e. } M = \bar{M}$$

$$A = \bar{A}$$

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§5.4 The Itô formula

Defn. 5.4.1 A continuous semi-martingale is a process X having

$$X(t) = X_0 + M(t) + A(t)$$

$$\begin{array}{c} \uparrow \\ \text{Prog. meas.} \end{array} \quad \begin{array}{c} \uparrow \\ M^{\text{c},0} \end{array} \quad \begin{array}{c} \uparrow \\ FV^{\text{c},0} \end{array}$$

$$\underline{SM}^{\text{c}}(\{\mathcal{F}_t\}, P)$$

$$\underline{M}_{\text{loc}}^{\text{c}} \subset \underline{SM}^{\text{c}} \quad \underline{FV}^{\text{c}} \subset \underline{SM}^{\text{c}}$$

Remark 5.4.5. Given $X, Y \in \underline{SM}^{\text{c}}$, $X = X_0 + M + A$
 $Y = Y_0 + N + B$ where $M, N \in \underline{M}_{\text{loc}}^{\text{c},0}$, $A \otimes B \in \underline{FV}^{\text{c},0}$

Define $[X, Y] \stackrel{\Delta}{=} [M, N]$

$\overbrace{\quad \quad \quad}^{\text{Def. 4.7.21.}}$

prop. 5.4.6. Given $X, Y, Z \in \underline{SM}^{\text{c}}$, $\alpha, \beta \in \mathbb{R}$. we have

(i) $[X] = [X, X]$

(ii) $[X, Y] = [Y, X]$

(iii) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$

(iv) If $X \in \underline{FV}^{\text{c}}$, then $[X, Y] = 0$

(v) $X \in \underline{FV}^{\text{c}} \Leftrightarrow [X] = 0$

Defn. 5.4.8 & 5.4.10:

suppose $X \in \underline{SM}^{\text{c}}(\{\mathcal{F}_t\}, P)$ has the form

$$X = X_0 + M + A \text{ for (necessarily unique) } M \in \underline{M}_{\text{loc}}^{\text{c},0}, A \in \underline{FV}^{\text{c}}$$

Then $L_{\text{loc}}^2(X, \{\mathcal{F}_t\}, P)$ denote the set of all processes $\{\Phi(t)\}$ st

(a) $(\Phi(t), \mathcal{F}_t)$ is prog. meas.

(b) $\int_0^t |\Phi(s)|^2 d[M]_s < \infty$ a.s. for each $t \in [0, \infty)$

(c) $\int_0^t |\Phi(s)| dA_s < \infty$ a.s. for each $t \in [0, \infty)$

Thus $\Phi \circ M$ exists in $\underline{M}_{\text{loc}}^{\text{c},0}$ (by defn. 5.3.15)

$\Phi \circ A$ exists in $\underline{FV}^{\text{c},0}$ (by defn. 5.2.10)

$$\int_0^t \Phi(s) dA_s$$

Remark

$$\text{We then define } \Phi \circ X \stackrel{\Delta}{=} \underbrace{\Phi \circ M}_{M_{loc}^{C,0}} + \underbrace{\int_0^{\cdot} \Phi(s) dA(s)}_{FV^{C,C}}$$

i.e. $\Phi \circ X \in \mathbb{S}\mathbb{M}^{C,0}$

Remark 5.4.17: Suppose $X \in \mathbb{S}\mathbb{M}^C(\{\mathcal{F}_t\}, P)$

Suppose that $(\Phi(t), \mathcal{F}_t)$ is continuous (i.e. $S \mapsto \Phi(S, w)$ is continuous for each w) and adapted.

i.e. $\{\Phi(t), \mathcal{F}_t\}$ is progressively measurable.

Fix $t \in [0, \infty)$, then $S \mapsto \Phi(S, w)$ is uniformly bounded over $0 \leq S \leq t$ i.e. we have $|\Phi(S, w)| \leq \bar{\gamma} < \infty$ for all $0 \leq S \leq t$,

$$\begin{aligned} \text{Then } \int_0^t |\Phi(s)|^2 d[M](s, w) &\leq \int_0^t \bar{\gamma}^2 d[M](s, w) \\ &\leq \bar{\gamma}^2 \underbrace{\int_0^t d[M](s, w)}_{[M](t, w)} \\ &\leq \bar{\gamma}^2 [M](t, w) < \infty \end{aligned}$$

i.e. $\Phi \circ X$ exists in $\mathbb{S}\mathbb{M}^{C,0}$ whenever $(\Phi(t), \mathcal{F}_t)$ is continuous and adapted.

If $X \& Y \in \mathbb{S}\mathbb{M}^C$ Then the integrals $X \circ Y$ and $Y \circ X$ exist in $\mathbb{S}\mathbb{M}^{C,0}$.

Recall Thm 5.1.15: If $A, B: [0, \infty) \rightarrow \mathbb{R}$ are continuous with lbV , then

$$A(t)B(t) = A(0)B(0) + \int_0^t A(s)dB(s) + \int_0^t B(s)dA(s)$$

Thm 5.4.19 (Itô integration-by-parts)

Suppose $X \& Y \in \mathbb{S}\mathbb{M}^C(\{\mathcal{F}_t\}, P)$, Then

$$\begin{aligned} X(t)Y(t) &= X(0)Y(0) + \int_0^t X(s)dY(s) + \int_0^t Y(s)dX(s) \\ &+ [X, Y](t) \end{aligned}$$

$(X \circ Y)(t) \quad (Y \circ X)(t)$

Terminology A function $F: \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be of class C^2 (or a C^2 -function) when the partial derivatives $\frac{\partial F}{\partial x_i}$ and $\frac{\partial^2 F}{\partial x_i \partial x_j}$ exist and are continuous on \mathbb{R}^d .

Theorem: 5.4.27: Suppose $x_1, x_2, \dots, x_d \in \mathbb{S}^M$ and $F: \mathbb{R}^d \rightarrow \mathbb{R}$ is a C^2 -function, then

$$F(x_1(t), x_2(t), \dots, x_d(t)) = F(x_1(0), \dots, x_d(0)) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(x_1(s), x_2(s), \dots, x_d(s)) dx_i(s) + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(x_1(s), \dots, x_d(s)) d[x_i, x_j](s)$$

A function $F: [0, \infty) \otimes \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be of class $C^{1,2}$ (or a $C^{1,2}$ -function) when the partial derivatives $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x_i}, \frac{\partial^2 F}{\partial x_i \partial x_j}$ exist and are continuous on $[0, \infty) \times \mathbb{R}^d$.

Theorem 5.4.27 (slightly generalized): suppose $x_1, \dots, x_d \in \mathbb{S}^M$ and $F: [0, \infty) \otimes \mathbb{R}^d \rightarrow \mathbb{R}$ is a $C^{1,2}$ -function, then

$$F(t, x_1(t), \dots, x_d(t)) = F(0, x_1(0), \dots, x_d(0)) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(s, x_1(s), \dots, x_d(s)) + \int_0^t \frac{\partial F}{\partial t}(s, x_1(s), \dots, x_d(s)) ds + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(s, x_1(s), \dots, x_d(s)) d[x_i, x_j](s)$$

Let $\{w(t)\}$ be a scalar Wiener process, let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a C^2 -func. then

$$f(w(t)) = f(0) + \int_0^t (Df)(w(s)) dw(s) + \frac{1}{2} \int_0^t (D^2f)(w(s)) d[w, w](s)$$

Ex. Evaluate (1) $\int_0^t w(s) dw(s)$

$$(2) \int_0^t \cos(w(s)) dw(s)$$

as pathwise Lebesgue integrals.

$$\text{Let } f(x) \stackrel{\Delta}{=} \frac{x^2}{2} \quad Df(x) = x \quad D^2f(x) = 1$$

$$\frac{1}{2} w^2(t) = c + \int_0^t w(s) dw(s) + \frac{1}{2} \int_0^t 1 \cdot ds$$

$$\therefore \int_0^t w(s) dw(s) = \frac{w^2(t)}{2} - \frac{t}{2}$$

$$(1) \quad f(x) = \sin(x) \quad Df(x) = \cos(x) \quad D^2f(x) = -\sin(x)$$

$$\therefore \sin(w(t)) = \sin(0) + \int_0^t \cos(w(s)) dw(s) + \frac{1}{2} \int_0^t -\sin(w(s)) ds$$

$$\therefore \int_0^t \cos(w(s)) dw(s) = \sin(w(t)) + \frac{1}{2} \int_0^t \sin(w(s)) ds$$

Evaluate $E[\cos^2(x)]$ where $x \sim N(0, 1)$

$$w(1) \sim N(0, 1)$$

$$E[\cos^2(x)] = E[\cos^2(w(1))]$$

we evaluate $\cos^2(w(t))$ by Zto formula.

$$F(x) = \cos^2(x)$$

$$Df(x) = -2\cos(x)\sin(x) = -\sin 2x$$

$$D^2f(x) = +2\sin^2(x) - 2\cos^2(x)$$

$$\text{By Zto} \quad (\cos^2(w(t))) = 1 + \int_0^t \overbrace{Df(w(s))}^{M(t)} dw(s) + \frac{1}{2} 2 \int_0^t \left\{ \sin^2(w(s)) - \cos^2(w(s)) \right\} ds$$

$$M \text{ is a } \in M_{loc}^{C,0} \quad |Df(w(s))| \leq 2$$

$$\text{By corollary 5-3.41. have that } M \in M_2^{C,0} \quad (1)$$

$$\therefore \mathbb{E}[M(t)] = \mathbb{E}[M(0)] = 0 \quad \text{--- (2)}$$

From (1) and (2), take expectations of (1) and use (2)

$$\mathbb{E}[\cos^2(w(t))] = 1 + 0 + \mathbb{E}\left[\int_0^t (\sin^2(w(s)) - \cos^2(w(s))) ds\right]$$

$$\sin^2(x) + \cos^2(x) = 1 \quad \text{for all } x$$

$$\text{Therefore } \mathbb{E}[\cos^2(w(t))] = 1 + \mathbb{E}\left[\int_0^t (1 - 2\cos^2(w(s))) ds\right]$$

↑
Fubini

$$= 1 + \int_0^t (1 - 2\mathbb{E}[\cos^2(w(s))]) ds \quad \text{--- (3)}$$

$$\text{Put } \varphi(t) = \mathbb{E}[\cos^2(w(t))] \quad \text{--- (4)}$$

$$\varphi(t) = 1 + t - 2 \int_0^t \varphi(s) ds$$

$$\therefore \dot{\varphi}(t) = 1 - 2\varphi(t) \quad \varphi(0) = 1$$

$$\therefore \varphi(t) = \frac{1 + e^{-2t}}{2}$$

$$\mathbb{E}[\cos^2(x)] = \mathbb{E}[\cos^2(w(1))] = \varphi(1) = \frac{1 + e^{-2}}{2}$$

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§ 5.5 Exponential semi martingales:

$$\exp[f(t)]$$

Defn. 5.5.1 Given $X \in \mathcal{SM}^c$, Define

$$\mathcal{E}(x)(t) \triangleq \exp\left\{x(t) - \frac{1}{2} [x](t)\right\}$$

\uparrow
Itô exp of X

prop. 5.5.2: Given $X \in \mathcal{SM}^c$, then

(a) $\mathcal{E}(x) \in \mathcal{SM}^c$

(b) $\mathcal{E}(x)(t) = 1 + \int_0^t \underbrace{\mathcal{E}(x)(s)}_{\mathcal{E}(x) \cdot X} dX(s)$

(c) If $X \in \underline{M}_{loc}^c$ then $\Sigma(X) \in \underline{M}_{loc}^c$

Read (a) (b)

(c) Recall if $X \in \underline{FV}^c$ then $\int_0^{\cdot} X dx \in \underline{FV}^{c,0}$

If $X \in \underline{M}_{loc}^c$, then $\Phi \cdot X \in \underline{M}_{loc}^{c,0}$

If $X \in \underline{SM}^c$, then $\Phi \cdot X \in \underline{SM}^{c,0}$

\therefore we see that $\Sigma(X) \cdot X \in \underline{M}_{loc}^{c,0}$.

from (b) get $\Sigma(X) \in \underline{M}_{loc}^c$ □

Question: Suppose that $X \in \underline{M}_{loc}^c$, then we know that

$\Sigma(X) \in \underline{M}_{loc}^c$, what additional conditions on X ensure that $\Sigma(X) \in \underline{M}^c$?

Theorem 5.5.10: suppose $X \in \underline{M}_{loc}^c$, put $[X]_{[0, \infty)} =$

$\lim_{t \rightarrow \infty} [X](t)$ if F_∞ -meas.

$\mathbb{E}[\exp(\frac{1}{2}[X](\infty))] < \infty$, uniformly integrable

then $(\Sigma(X)(t), \mathcal{F}_t)$ is a UI Martingale.

Corollary 5.5.11: suppose $X \in \underline{M}_{loc}^c (\{\mathcal{F}_t\}, P)$

If $\mathbb{E}[\exp(\frac{1}{2}[X](t))] < \infty$ for all $t \in [0, \infty)$

Then $(\Sigma(X)(t), \mathcal{F}_t)$ is a martingale.

$[w](t) = t$

$\mathbb{E}[\exp(\frac{1}{2}[w](t))] = e^{t/2} < \infty$ for $t \in [0, \infty)$

$\therefore (\Sigma(w), \mathcal{F}_t)$ is a martingale.

Yen kia An

§5.6 Levy Characterization of a Wiener process.

Recall Remark 4.7.25;

If $(W(t), \mathcal{F}_t)$ is a standard scalar w-proc, then

$w \in \underline{M}_{loc}^{c,0} (\{\mathcal{F}_t\})$ and $[w](t) = t$

Levy Theorem is essentially a converse mainly:

If $W \in M_{loc}^{C,0}(\{\mathcal{F}_t\}, P)$ s.t. $[W](t) = t$, then
 W is a standard N -proc.

To establish this, we need the following elementary 901-type result.

Lemma 5.6.1. Suppose (i) X is an \mathbb{R}^d -valued random vector on (Ω, \mathcal{F}, P)

(ii) $\mathcal{G} \subset \mathcal{F}$ is a σ -alg positive semidefinite
 (iii) a $d \times d$ symmetric non-negative definite matrix Q

i.e. $\theta' Q \theta \geq 0$ for all $\theta \in \mathbb{R}^d$

If for $\theta \in \mathbb{R}^d$, we have

$$\exp\{\imath \langle \theta, X \rangle\} | \mathcal{G} = \exp[-\frac{1}{2} \theta' Q \theta] \text{ a.s.}$$

then (i) $X \sim N(0, Q)$ i.e. $E(X) = 0$ $\text{cov}(X) = Q$

(ii) $X \perp \mathcal{G}$.

$$\text{Here } \langle \theta, X \rangle(w) \triangleq \sum_{i=1}^d \theta_i X_i(w)$$

$$\exp\{\imath \langle \theta, X \rangle\} = \cos(\langle \theta, X \rangle) + \imath \sin(\langle \theta, X \rangle)$$

$$\mathbb{E}[\exp\{\imath \langle \theta, X \rangle\} | \mathcal{G}] = \mathbb{E}[\cos(\langle \theta, X \rangle) | \mathcal{G}] + \imath \mathbb{E}[\sin(\langle \theta, X \rangle) | \mathcal{G}]$$

$\mathbb{E}[\exp(i\theta X)]$ characteristic function

Thm 5.6.2: If $W \in M_{loc}^{C,0}(\{\mathcal{F}_t\}, P)$ with $[W](t) = t$

then $(W(t), \mathcal{F}_t)$ is a scalar Wiener process

Proof: For each $\theta \in \mathbb{R}$. define

$$Z^\theta(t, \omega) \triangleq \exp[i\theta W(t, \omega) + \frac{1}{2} \theta^2 t] \quad \dots \quad (1)$$

Clearly, $(Z^\theta(t), \mathcal{F}_t)$ is a continuous process.

Suppose we can show that $Z^\theta \in M_{loc}^C(\{\mathcal{F}_t\}, P)$ for each θ adapted

$$\theta \in \mathbb{R} \quad \text{---} \quad (2)$$

then for each $\theta \in \mathbb{R}$, we have the following when
 $0 \leq s < t < \infty$

$$\mathbb{E}[Z^\theta(t) | \mathcal{F}_s] = Z^\theta(s) \text{ a.s.} \quad \text{---} \quad (3)$$

From (1) (3)

$$\mathbb{E}[\exp\{i\theta W(t) + \frac{1}{2}\theta^2 t\} | \mathcal{F}_s]$$

$$= \exp\{i\theta W(s) + \frac{1}{2}\theta^2 s\}$$

i.e.: $\mathbb{E}[\exp\{\overbrace{i\theta(W(t)-W(s))}^X\} | \mathcal{F}_s]$

$$= \exp[-\frac{1}{2}\theta^2(t-s)] \text{ a.s.} \quad \text{---} \quad (4)$$

From (4) and Lemma 5.6.1. get

$$W(t) - W(s) \sim N(0, t-s)$$

$$W(t) - W(s) \perp \mathcal{F}_s$$

i.e. $(W(t), \mathcal{F}_t)$ is a standard Wiener process.

It remains to show that (2) holds: *Def*

Fix some $\theta \in \mathbb{R}$ and define

$$f^\theta(t, x) = \exp[i\theta x + \frac{1}{2}\theta^2 t] \quad t \in [0, \infty), x \in \mathbb{R}$$

$$\begin{aligned} Z^\theta(t) &= f^\theta(t, W(t)) \\ &= f^\theta(t, 0) + \int_0^t \frac{\partial f}{\partial t}(s, W(s)) ds + \int_0^t \frac{\partial f}{\partial x}(s, W(s)) \cdot dW(s) \\ &= 1 + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, W(s)) ds \end{aligned}$$

$$\text{i.e. } Z^\theta(t) = 1 + \underbrace{\int_0^t \left(\frac{\partial f}{\partial x}(s, W(s)) \right) dW(s)}_{\in M^{loc} \sim \mathcal{C}^0}$$

i.e. $\mathbb{Z}^0 \in \mathcal{M}_{loc}^{c,0}(\{\mathcal{F}_t\}, P)$

Using the boundedness properties of $\frac{\partial F}{\partial x}$.

We can actually show that ② holds ✓

Multi dimensional version of Thm 5.6.2:

Suppose that $w^k \in \mathcal{M}_{loc}^{c,0}(\{\mathcal{F}_t\}, P) \quad k=1, 2, \dots, d$

st. $[w^k, w^j](t) = t \delta_{jk} \quad k, j = 1, \dots, d$

then $(w(t), \mathcal{F}_t)$ is an \mathbb{R}^d -valued standard Wiener process where

$$w(t) \triangleq (w^1(t), w^2(t), \dots, w^d(t))$$

§ 1.2. The Radon-Nikodym Thm.

Defn. 1.2.28. Given measures ν and μ on a meas. Space (E, \mathcal{S}) , then we say that ν is absolutely continuous wrt. μ when

$$A \in \mathcal{S} \text{ st } \mu(A)=0 \Rightarrow \nu(A)=0$$

Denote $\nu \ll \mu [\mathcal{S}]$

Corollary 1.2.13. suppose that (E, \mathcal{S}, μ) is a measure space.

and $f: E \rightarrow [0, \infty]$ is \mathcal{S} -meas.

For each $A \in \mathcal{S}$, define

$$\nu(A) \triangleq \int_A f \cdot d\nu \equiv \int_E \mathbf{1}_A f \cdot d\nu$$

By MCT. we see that

ν is also a measure on (E, \mathcal{S})

Moreover, if $\mu(A)=0$, then $\nu(A)=0$

i.e. $\nu \ll \mu [\mathcal{S}]$

John von Neumann

Theorem 1.3.20. suppose that ν and μ are finite measures on the measurable space (E, \mathcal{S}) ($\text{ie. } \mu(E) < \infty$
 $\nu(E) < \infty$)

could be extended
 / to σ -finite

If $V \ll \mu[\mathcal{S}]$, then there exists an \mathcal{H} -meas
fun. $f : E \rightarrow [0, \infty]$
st. $V(A) = \int_A f d\mu \quad A \in \mathcal{S}$.

Moreover, If $\tilde{f} : E \rightarrow [0, \infty]$ is \mathcal{S} -meas. and

$$V(A) = \int_A \tilde{f} d\mu \quad A \in \mathcal{S}.$$

then $f = \tilde{f}$ ae. wrt. $[\mu]$

↑
not necessarily wrt V !

Mar 25, Cont.

From ① and the chain rule for L -integrals (see Thm 1.2.17)

We have: if $g : E \rightarrow [0, \infty]$ is \mathcal{S} -meas.

$$\int_E g \cdot dv = \int_E g \cdot \left(\frac{dv}{d\mu} \right) \cdot d\mu$$

Replacing g by $\mathbb{1}_A g$ (for some $A \in \mathcal{S}$)

gives

$$\int_A g dv = \int_A g \left(\frac{dv}{d\mu} \right) d\mu, \text{ for each } A \in \mathcal{S}.$$

Remark 1.2.34: Given finite measure V and μ on (E, \mathcal{S}) st.
 $V \ll \mu[\mathcal{S}]$, given a σ -algebra $\mathcal{H} \subset \mathcal{S}$.

Define $V_H(A) \triangleq V(A) \quad A \in \mathcal{H}$

$\mu_H(A) \triangleq \mu(A) \quad A \in \mathcal{H}$

Then V_H and μ_H are finite measures on (E, \mathcal{H}) , and

Clearly $V_H \ll \mu_H[\mathcal{H}]$

By Thm 1.2.30, \exists some \mathcal{H} -meas. $f_H : E \rightarrow [0, \infty]$ st.

$$V_H(A) = \int_A f_H d\mu_H \text{ for all } A \in \mathcal{H}$$

i.e. $\underbrace{V_H(A)}_{V(A)} = \int_A f_H d\mu_H \quad A \in \mathcal{H}$
 $\qquad \qquad \qquad \text{H-meas.}$

i.e. $= \int_A f_H \cdot d\mu$

$$V(A) = \int_A f_H \cdot d\mu \quad \text{for } A \in \mathcal{H}$$

\uparrow $\mathcal{H}\text{-meas.}$

We will denote f_H by $(\frac{dV}{d\mu} |_{\mathcal{H}})$

i.e. when $V \ll \mu[\mathcal{S}]$ and $\mathcal{H} \subset \mathcal{S}$ is a σ -alg.

then \exists a μ -ae unique \mathcal{H} -meas. function $\frac{dV}{d\mu} |_{\mathcal{H}}$

$E \rightarrow [0, +\infty]$ st.

$$V(A) = \int_A (\frac{dV}{d\mu} |_{\mathcal{H}}) d\mu \quad \text{for all } A \in \mathcal{H}$$

i.e. " $dV(A) = (\frac{dV}{d\mu} |_{\mathcal{H}}) d\mu(A)$ for all $A \in \mathcal{H}$ ". ②

z.e. when $g: E \rightarrow [0, \infty]$ is \mathcal{H} -meas. then we see from

②, and the chain rule for L -integrals (cf Thm 1.2.14)

$$\int_A g dV = \int_A g(\underbrace{\frac{dV}{d\mu} |_{\mathcal{H}}}_{\mathcal{H}\text{-meas.}}) d\mu \quad \text{for } A \in \mathcal{H}.$$

for each \mathcal{H} -meas. $g: E \rightarrow [0, \infty]$

Defn: 1.2.35: Given the measures V and μ on (E, \mathcal{S}) we write

$V \equiv \mu[\mathcal{S}]$ to indicate that

$V \ll \mu[\mathcal{S}]$ and $\mu \ll V[\mathcal{S}]$

i.e. for each $A \in \mathcal{S}$, we have that

$V(A) = 0$ iff $\mu(A) = 0$

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Thm. 1.2.30 (R-N Thm)

Given ν meas. ν and μ on (E, \mathcal{S}) st. $\nu \ll \mu[\mathcal{S}]$

(i.e. if $A \in \mathcal{S}$, st. $\mu(A) = 0$, then $\nu(A) = 0$)

Then \exists some \mathcal{S} -meas. $f: E \rightarrow [0, \infty]$

$$\text{st. } \nu(A) = \int_A f d\mu \quad A \in \mathcal{S}$$

Moreover, f is unique μ -a.e.

For an arbitrary but fixed choice of a function f given by Thm 1.2.30, we will write

$$\left(\frac{d\nu}{d\mu} \right)$$

a R-N derivative of ν wrt. μ .

$$\text{i.e. } \nu(A) = \int_A \left(\frac{d\nu}{d\mu} \right) d\mu \text{ for all } A \in \mathcal{S}.$$

$$\text{i.e. } d\nu(A) = \left(\frac{d\nu}{d\mu} \right) d\mu | A \text{ for all } A \in \mathcal{S}''$$

Mar 25, 2009. (Cont.)

We say that ν and μ are equivalent measures on \mathcal{S} .

Defn. 1.2.36. Given a measure space (E, \mathcal{S}, μ) and a \mathcal{S} -meas.

function $f: E \rightarrow [0, \infty]$

We say that f is μ -strictly positive when
 $f > 0 \text{ } \mu\text{-a.e.}$

i.e. $\mu \{ x \in E : f(x) = 0 \} = 0$

prop. 1.2.37: Given finite measures ν and μ on (E, \mathcal{S}) st $\nu \ll \mu[\mathcal{S}]$, then

$$(a) \frac{d\nu}{d\mu} > 0 \text{ } \mu\text{-a.e.}$$

i.e. $\nu \{ x \in E : \frac{d\nu}{d\mu}(x) = 0 \} = 0$

but we can easily have $\mu \{ x \in E : \frac{d\nu}{d\mu}(x) = 0 \} \geq 0$ (!!)

$$(b) v \equiv u[\xi], \Leftrightarrow \frac{dv}{du} > 0 \quad u-a \in \mathcal{X} \quad v-a \in \mathcal{Y}$$

i.e. $u \setminus \{x \in \mathcal{X} : \frac{dv}{du}(x) = 0\} = \emptyset$

(c) if $v \equiv u[\xi]$, then

$$\frac{dv}{du} = \left(\frac{du}{dv} \right)^{-1} \quad a \in \text{wrt. both } v \text{ and } u$$

1.4.B Appl. of Radon Nikodym Thm.

Remark 1.4.18: Given a prob. space (Ω, \mathcal{F}, P)

$\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}$ are σ -algs.

a prob. meas. \tilde{P} on (Ω, \mathcal{F}) st.

$$\tilde{P} \ll P[\mathcal{F}]$$

$$\text{Put } X_{\mathcal{G}} \triangleq \frac{d\tilde{P}}{dP} \Big|_{\mathcal{G}}$$

i.e. $X_{\mathcal{G}}$ is \mathcal{G} -meas. and $\tilde{P}(A) = \int_A X_{\mathcal{G}} \cdot dP \quad A \in \mathcal{G}$
 $= E[X_{\mathcal{G}}; A] \quad A \in \mathcal{G} \quad \dots \quad (1)$

Similarly, for $X_{\mathcal{H}} \triangleq \frac{d\tilde{P}}{dP} \Big|_{\mathcal{H}}$

we have $\tilde{P}(A) = E[X_{\mathcal{H}}; A] \text{ for all } A \in \mathcal{H} \quad \dots \quad (2)$

from (1) and (2)

$$E[X_{\mathcal{G}}; A] = E[X_{\mathcal{H}}; A] \text{ for all } A \in \mathcal{G}$$

i.e. $E[X_{\mathcal{G}}; A] = E[X_{\mathcal{H}}; A] \quad A \in \mathcal{G}$

$$E[E[X_{\mathcal{H}}|\mathcal{G}]; A] = E[X_{\mathcal{G}}; A] \text{ for all } A \in \mathcal{G}$$

\uparrow \uparrow
 \mathcal{G} -meas. \mathcal{G} -meas

i.e. $X_{\mathcal{G}} = E[X_{\mathcal{H}}|\mathcal{G}] \quad P\text{-ae.}$

$$\text{i.e. } \frac{d\tilde{P}}{dP} \Big|_{\mathcal{F}_t} = \mathbb{E} \left[\frac{d\tilde{P}}{dP} \Big|_{\mathcal{F}_T} \mid \mathcal{F}_t \right] \text{ P-a.e.}$$

§5.7 The Girsanov Transformation

Question: $X \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, P)$, Replace the measure P with some other prob. measure \tilde{P} . Do we have that

$$X \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, \tilde{P}) ?$$

Ans: generally NOT

However, if $\tilde{P} = P[\mathcal{F}]$, then the Girsanov Theorem says that \exists some $A \in \mathcal{F}V^c(\{\mathcal{F}_t\})$ st.

$$(X - A) \in \mathcal{M}_{loc}^c(\{\mathcal{F}_t\}, \tilde{P})$$

Condition 5.7.1

Given a filtration $\{\mathcal{F}_t\}$ in the prob. space (Ω, \mathcal{F}, P) st.

\mathcal{F} includes all P -null events in \mathcal{F} , and given a prob. measure \tilde{P} on (Ω, \mathcal{F}) such that

$$\tilde{P} = P[\mathcal{F}] \quad (\text{Advanced result, } \tilde{P} \ll P[\mathcal{F}])$$

$$\text{Define } \Lambda(t) \triangleq \frac{d\tilde{P}}{dP} \Big|_{\mathcal{F}_t}, \quad t \in [0, \infty]$$

↑
note

From Remark 4.18.

$$\Lambda(t) = \mathbb{E}[\Lambda(\infty) \mid \mathcal{F}_t] \text{ as. for all } t \in [0, \infty]$$

i.e. $\Lambda \in \mathcal{M}(\{\mathcal{F}_t\}, P)$ ↑
in note P-martingale

call Λ the density process of \tilde{P} wrt. P

Condition 5.7.4. We will assume

$$\Lambda \in \mathcal{M}^c(\{\mathcal{F}_t\}, P)$$

Suppose that $(X(t), \mathcal{F}_t)$ is an adapted process,

Fix some $t \in [0, \infty)$ $A \in \mathcal{F}_t$, Then

$$\tilde{\mathbb{E}}[X(t); A] = \int_A X(t) d\tilde{P} = \int_A X(t) \left(\frac{d\tilde{P}}{dP} \right) dP$$

(Remark 1.2.33 Note X_t is \mathcal{F}_t -meas. $A \in \mathcal{F}_t$)

i.e. $\tilde{\mathbb{E}}[X(t); A] = \int_A X(t) \Lambda(t) dP$

If $\tilde{\mathbb{E}}[X(t); A] = \mathbb{E}[X(t)\Lambda(t); A]$ for all $A \in \mathcal{F}_t$