

A short coming of Riemann integration

Example:  $X_{\mathbb{Q}}(t) = \begin{cases} 1 & t \in \mathbb{Q} \text{ rational} \\ 0 & t \notin \mathbb{Q} \end{cases}$   $X_{\mathbb{Q}} : \mathbb{R} \rightarrow \mathbb{R}$

show that  $X_{\mathbb{Q}}(t)$  is NOT Riemann integrable on  $[0, 1]$   
 proof: Let  $\delta > 0$ , and  $\mathcal{P}$  be a partition of  $[0, 1]$  with  $L(\mathcal{P}) < \delta$

$$= \{0 = x_0 < x_1 < \dots < x_n = 1\}$$

we note that  $[0, 1] \cap \mathbb{Q}$  is dense in  $[0, 1]$ ,  
 and  $[0, 1] \setminus \mathbb{Q}$  is dense in  $(0, 1)$  Hence we can find Riemann sums

$$S_{\mathbb{R}}(f, \mathcal{P}) = \sum_{i=1}^n f(\gamma_i) (x_i - x_{i-1}) \quad \gamma_i \in [x_{i-1}, x_i] \cap \mathbb{Q}$$

such  $\gamma_i$  always exists, by density)

$$S_{\mathbb{I}}(f, \mathcal{P}) = \sum_{i=1}^n f(\gamma_i) (x_i - x_{i-1}) \quad \gamma_i \in [x_{i-1}, x_i] \setminus \mathbb{Q}$$

we have  $S_{\mathbb{R}}(X_{\mathbb{Q}}, \mathcal{P}) = \sum_{i=1}^n (x_i - x_{i-1}) = x_1 - x_0 + x_2 - x_1 + \dots + x_n - x_{n-1}$   
 $= -x_0 + x_n = -0 + 1 = 1$

whereas  $S_{\mathbb{I}}(X_{\mathbb{Q}}, \mathcal{P}) = \sum_{i=1}^n 0 (x_i - x_{i-1}) = 0$

Thus if  $\epsilon > 0$ , say  $1 \geq \epsilon$ , then for any  $\delta > 0$ , no matter how small

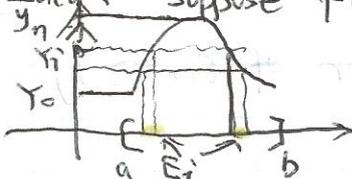
we can get Riemann sums, as above, with

$$|S_{\mathbb{R}}(X_{\mathbb{Q}}, \mathcal{P}) - S_{\mathbb{I}}(X_{\mathbb{Q}}, \mathcal{P})| = 1 \geq \epsilon.$$

Thus, that definition of Riemann integrability cannot be satisfied  $\square$

Lebesgue measure and integral

Idea: suppose  $f : [a, b] \rightarrow \mathbb{R}$   $f(t) \geq 0$  for  $t \in [a, b]$



Instead of dividing domain into intervals.

divide the range

Let  $y_0 = \inf_{t \in [a, b]} f(t)$   $Y_n = \sup_{t \in [a, b]} f(t) < \infty$

suppose

choose  $y_0 < y_1 < y_2 < \dots < y_{i-1} < y_i < \dots < y_n$

$\underbrace{\hspace{10em}}_{i\text{-th division interval}}$

Let  $E_i = \{t \in [a, b] : y_{i-1} \leq f(t) < y_i\}$

clearly  $E_i \cap E_j = \emptyset$  if  $i \neq j$  and  $[a, b] = \bigcup_{i=1}^n E_i \cup \{t \in [a, b] : f(t) = y_n\}$

We require a notion of the "length"  $\lambda(E_i)$  of each set  $E_i$ .  
Instead of a Riemann sum, a sum of the form

$$\sum_{i=1}^n \gamma_i \lambda(E_i) \text{ will approximate the integral}$$

First goal: Define  $\lambda(E)$  for "appropriate" set  $E \subset \mathbb{R}$

Construction of the "length function"  $\lambda$ :

Step # 1: Length of an interval = if  $a < b$ , then

$$L((a, b)) = b - a$$

$$L((a, \infty)) = \infty \quad L((-\infty, b)) = \infty$$

Step # 2: Define the Lebesgue outer measure

Let  $E \subset \mathbb{R}$ . A sequence  $\{I_n\}_{n=1}^{\infty}$  of open intervals.  
called a cover of  $E$  if  $E \subset \bigcup_{n=1}^{\infty} I_n$

Note, we have that  $I_n$  is of the form  $(a_n, b_n)$   $a_n < b_n$   
or  $(a, \infty)$  or  $(-\infty, b)$  or  $\emptyset = (b_n, a_n)$   $a_n \leq b_n$ ,

a notation which allows finite covers as well.

The Lebesgue measure of  $E$  is given by

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : \{I_n\}_{n=1}^{\infty} \text{ is a seq. of open intervals which covers } E \right\}$$

Proposition: Let  $a < b$  in  $\mathbb{R}$ , then if

$J$  denotes any of the intervals  $(a, b)$ ,  $(a, b]$ ,  $[a, b)$ ,  $[a, b]$

We have  $\lambda^*(J) = b - a$

Proof: Let  $\varepsilon > 0$ , and fix  $J = (a, b]$  proofs for other types of interval are similar.

First, note that  $\{(a, b + \frac{\varepsilon}{2})\}$  is a cover of  $(a, b]$  by a "sequence" of open intervals. Hence

$$\lambda^*((a, b]) = \inf \left\{ \sum_{i=1}^{\infty} l(I_i) : \{I_n\}_{n=1}^{\infty} \text{ is a cover of } (a, b] \text{ by open intervals} \right\}$$

$$\leq l((a, b + \frac{\varepsilon}{2})) = b + \frac{\varepsilon}{2} - a = b - a + \frac{\varepsilon}{2} < b - a + \varepsilon$$

since  $\varepsilon > 0$  can be chosen arbitrarily.

$$\lambda^*((a, b]) \leq b - a$$

Now suppose  $\{I_n\}_{n=1}^{\infty}$  is any cover of  $[a, b]$  by open intervals. First, note that  $[a + \frac{\epsilon}{2}, b]$  (if  $\frac{\epsilon}{2} < b - a$ ) a compact subset of  $[a, b]$  and is covered by  $\{I_n\}_{n=1}^{\infty}$ . By compactness, there is a finite subsequence  $I_{n_1}, I_{n_2}, \dots, I_{n_k}$  st  $[a + \frac{\epsilon}{2}, b] \subset \bigcup_{i=1}^k I_{n_i}$

Let  $I_{n_i} = (c_i, d_i)$  for each  $i$ . We may suppose, by reordering the indices, dropping intervals which are contained wholly within others and truncating in finite intervals, that

$$\begin{aligned} c_1 < a + \frac{\epsilon}{2}, \quad b < d_k \quad \text{and} \quad c_{i+1} < d_i \quad \text{for} \quad i=1, \dots, k-1 \\ \sum_{n=1}^{\infty} l(I_n) &\geq \sum_{i=1}^k l(I_{n_i}) = \sum_{i=1}^k (d_i - c_i) = \underbrace{d_1 - c_1}_{>0} + \underbrace{d_2 - c_2}_{>0} + \dots + \underbrace{d_{k-1} - c_{k-1}}_{>0} + d_k \\ &= -c_1 + \underbrace{d_1 - c_2}_{>0} + \underbrace{d_2 - c_3}_{>0} + \dots + \underbrace{d_{k-1} - c_k}_{>0} + d_k \\ &> -c_1 + d_k = d_k - c_1 \geq b - (a + \frac{\epsilon}{2}) = b - a - \frac{\epsilon}{2} \end{aligned}$$

$$\begin{aligned} \text{Thus} \quad \lambda^*(a, b) &= \inf \left\{ \sum_{n=1}^{\infty} l(I_n) \mid \{I_n\}_{n=1}^{\infty} \text{ cover of } [a, b] \text{ by open intervals} \right\} \\ &\geq b - a - \frac{\epsilon}{2} \end{aligned}$$

since  $\epsilon$  can be chosen arbitrarily, we conclude that  $\lambda^*(a, b) \geq b - a$

Problems with outer measure. It is possible to have  $E, F \subset \mathbb{R}$ ,  $E \cap F = \emptyset$ , but  $\lambda^*(E \cup F) < \lambda^*(E) + \lambda^*(F)$

We will get to this by an example later.

step # 3: Define (Lebesgue) measurable sets

We define  $E \subset \mathbb{R}$  to be measurable if for any set  $A \subset \mathbb{R}$ .

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \setminus E)$$

often also known as "Carathéodory's Criterion"

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Text: Information forthcoming

Lebesgue outer measure

$$E \subset \mathbb{R} \quad \lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subset \bigcup_{n=1}^{\infty} I_n \text{ each } I_n \text{ is an open interval} \right\}$$

Properties of  $\lambda^*$ :

(i)  $\lambda^*(\emptyset) = 0$

(ii)  $\lambda^*(E) \geq 0 \quad \forall E \subset \mathbb{R}$  (non-negativity)

(iii)  $\lambda^*(E) \leq \lambda^*(F)$  if  $E \subset F \subset \mathbb{R}$  (increasing)

(iv)  $\lambda^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \lambda^*(E_n)$   $E_n \subset \mathbb{R}$  for each  $n$   
 ( $\sigma$ -subadditivity) (many  $E_n$ 's may be  $\emptyset$  account for finite unions)

Proof: (i) (ii) obvious.

(iii) if  $\{I_n\}_{n=1}^{\infty}$  is a cover for  $F$ , then it is a cover for  $E$  too.

Hence  $\lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : \{I_n\}_{n=1}^{\infty} \text{ cover of } E \text{ by open intervals} \right\}$   
 $\leq \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : \dots \text{ of } F \right\}$  (inf over more covers)  
 $= \lambda^*(F)$

(iv) If  $\sum_{n=1}^{\infty} \lambda^*(E_n) = \infty$  we are done. Let  $\varepsilon > 0$

suppose  $\sum_{n=1}^{\infty} \lambda^*(E_n) < +\infty$ . Let for each  $n$ .

$\{I_{i,n}\}_{i=1}^{\infty}$  be a sequence of open intervals covering  $E_n$

such that  $\sum_{i=1}^{\infty} \ell(I_{i,n}) < \lambda^*(E_n) + \frac{\varepsilon}{2^n}$

Then  $\{I_{i,n}\}_{i=1, n=1}^{\infty, \infty}$  is a sequence of open intervals, covering

$$\bigcup_{n=1}^{\infty} E_n$$

We have  $\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{i,n}) \leq \sum_{n=1}^{\infty} \left( \lambda^*(E_n) + \frac{\varepsilon}{2^n} \right)$

$$= \left( \sum_{n=1}^{\infty} \lambda^*(E_n) \right) + \varepsilon$$

and hence  $\lambda^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{i,n}) \leq \sum_{n=1}^{\infty} \lambda^*(E_n) + \varepsilon$

and thus  $\lambda^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \lambda^*(E_n)$  as  $\varepsilon$  can be chosen arbitrarily

Recall  $A \subset \mathbb{R}$  is (Lebesgue) measurable if  $\square$

for any  $E \subset \mathbb{R}$ ,  $\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$

Notation:  $\mathcal{P}(\mathbb{R}) = \{E : E \subset \mathbb{R}\}$  "power set"

$\mathcal{L}(\mathbb{R}) = \{E \in \mathcal{P}(\mathbb{R}) : E \text{ is Lebesgue measurable}\}$

Theorem (properties of  $\mathcal{L}(\mathbb{R})$ )

- (i)  $\emptyset, \mathbb{R} \in \mathcal{L}(\mathbb{R})$
- (ii)  $A \in \mathcal{L}(\mathbb{R}) \Rightarrow A^c = \mathbb{R} \setminus A \in \mathcal{L}(\mathbb{R})$
- (iii)  $A_1, A_2, \dots \in \mathcal{L}(\mathbb{R}) \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{L}(\mathbb{R})$

If, moreover in (iii) we have  $A_i \cap A_j = \emptyset$  for  $i \neq j$ , then

$$\lambda^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda^*(A_n)$$

this det implies disjoint union

Let  $E \subset \mathbb{R}$

proof: (i)  $\lambda^*(E \cap \emptyset) + \lambda^*(E \setminus \emptyset) = \lambda^*(\emptyset) + \lambda^*(E) = \lambda^*(E)$   
 $\Rightarrow \emptyset \in \mathcal{L}(\mathbb{R})$

proof for  $\mathbb{R}$  is nearly identical

(ii) Let  $E \subset \mathbb{R}, A \in \mathcal{L}(\mathbb{R})$

$$\lambda^*(E \cap (\mathbb{R} \setminus A)) + \lambda^*(E \setminus (\mathbb{R} \setminus A)) = \lambda^*(E \setminus A) + \lambda^*(E \cap A) = \lambda^*(E)$$

by measurability of A

$$\Rightarrow \mathbb{R} \setminus A \in \mathcal{L}(\mathbb{R})$$

(iii) Let  $E \subset \mathbb{R}, A_1, A_2, \dots \in \mathcal{L}(\mathbb{R}) \quad A = \bigcup_{n=1}^{\infty} A_n$

First let's do a set computation

$$\begin{aligned} E \cap A &= E \cap \left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} (E \cap A_n) = (E \cap A_1) \cup (E \cap A_2) \cup \dots \\ &= (E \cap A_1) \cup ((E \setminus A_1) \cap A_2) \cup ((E \setminus (A_1 \cup A_2)) \cap A_3) \cup \dots \\ &= \bigcup_{n=1}^{\infty} \left( (E \setminus \bigcup_{i=1}^{n-1} A_i) \cap A_n \right) \end{aligned}$$

Now by  $\sigma$ -subadditivity we obtain

$$(f) \quad \lambda^*(E) \leq \lambda^*(E \setminus A) + \lambda^*(E \cap A) \leq \lambda^*(E \setminus A) + \sum_{n=1}^{\infty} \lambda^*( (E \setminus \bigcup_{i=1}^{n-1} A_i) \cap A_n )$$

we don't yet know if  $A \in \mathcal{L}(\mathbb{R})$

On the other hand we have, since each  $A_n$  is measurable

$$\begin{aligned} \lambda^*(E) &= \lambda^*(E \cap A_1) + \lambda^*(E \setminus A_1) \\ &= \lambda^*(E \cap A_1) + \lambda^*( (E \setminus A_1) \cap A_2 ) + \lambda^*( (E \setminus (A_1 \cup A_2)) \cap A_3 ) + \dots \end{aligned}$$

$$= \lambda^*(E \cap A_1) + \lambda^*((E \cap A_1) \cap A_2) + \lambda^*((E \cap (A_1 \cup A_2)) \cap A_3) \\ + \lambda^*(E \cap \bigcup_{i=1}^3 A_i)$$

$$= \sum_{i=1}^k \lambda^*( (E \cap \bigcup_{j=1}^{i-1} A_j) \cap A_i ) + \lambda^*(E \cap \bigcup_{i=1}^k A_i)$$

$$\geq \sum_{i=1}^k \lambda^*( (E \cap \bigcup_{j=1}^{i-1} A_j) \cap A_i ) + \lambda^*(E \cap \bigcup_{i=1}^{\infty} A_i)$$

↑  
"increasing" property of  $\lambda^*$

Letting  $k \rightarrow \infty$

$$(ff) \lambda^*(E) \geq \sum_{i=1}^{\infty} \lambda^*( (E \cap \bigcup_{j=1}^{i-1} A_j) \cap A_i ) + \lambda^*(E \cap \bigcup_{i=1}^{\infty} A_i)$$

combine (f) and (ff) - double daggers

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^c) \quad \text{Thus } A \in \mathcal{L}(\mathbb{R})$$

Now suppose  $A_i \cap A_j = \emptyset$  if  $i \neq j$ , then, for each  $A_i$

$$(E \cap \bigcup_{j=1}^i A_j) \cap A_i = A_i \cap E = A_i \quad (\text{if } A_i \subset E)$$

Thus letting  $E = A$  in (ff) we have

$$\lambda^*(A) \geq \sum_{i=1}^{\infty} \lambda^*(A \cap A_i) + \lambda^*(A \cap A^c)$$

$\underbrace{\qquad\qquad\qquad}_{=0}$

$$= \sum_{i=1}^{\infty} \lambda^*(A_i)$$

and  $\lambda^*(A) \leq \sum_{i=1}^{\infty} \lambda^*(A_i)$  holds, by  $\sigma$ -subadditivity.

Hence

$$\lambda^*(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \lambda^*(A_i) \quad \square$$

We consider  $\lambda^* = \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$

if  $A \in \mathcal{L}(\mathbb{R})$ , we let  $\lambda(A) = \lambda^*(A)$

Thus  $\lambda = \lambda^* |_{\mathcal{L}(\mathbb{R})}$  : restricted function  $\mathcal{L}(\mathbb{R}) \rightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$

Theorem (properties of  $\lambda$ )

- property (i)  $\lambda(\emptyset) = 0$
- (ii)  $\lambda(A) \geq 0$  for  $A \in \mathcal{L}(\mathbb{R})$  (nonnegativity)
- (iii)  $\lambda(A) \leq \lambda(B)$  if  $A \subset B, A, B \in \mathcal{L}(\mathbb{R})$  (increasing)
- (iv) if  $A_1, A_2, \dots \in \mathcal{L}(\mathbb{R})$  pairwise disjoint  $A_i \cap A_j = \emptyset$  for  $i \neq j$ , then  $\lambda(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \lambda(A_i)$   $\sigma$ -additivity

proof: collect results about  $\lambda^*$ ; restrict to  $\mathcal{L}(\mathbb{R})$

Note: (ii) (iv)  $\Rightarrow$  (iii)  $\square$

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- This week - open sets Borel sets Cantor set
- existence of non-measurable set

Extent of Lebesgue measurable sets: open sets, Borel sets.

$\mathcal{L}(\mathbb{R}) = \{A \subset \mathbb{R} : \lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A) \text{ for any } E \subset \mathbb{R}\}$

Goal: open sets are Lebesgue measurable.

Proposition: if  $a < b$  in  $\mathbb{R}$ , then the open interval  $(a, b) \in \mathcal{L}(\mathbb{R})$ .

proof: Let  $E \subset \mathbb{R}$  we wish to show

$\lambda^*(E) = \lambda^*(E \cap (a, b)) + \lambda^*(E \setminus (a, b))$

Note that " $\leq$ " is trivial, by  $\sigma$ -subadditivity of  $\lambda^*$

It remains to see " $\geq$ "  $E \subset \bigcup_{n=1}^{\infty} I_n$

If  $\lambda^*(E) = \infty$ , we are done. If  $\lambda^*(E) < \infty$ , Let  $\epsilon > 0$  and  $\{I_n\}_{n=1}^{\infty}$  be a sequence of open intervals for which

$\sum_{n=1}^{\infty} \ell(I_n) < \lambda^*(E) + \epsilon/2$

for each index  $n$ .

$L_n = I_n \cap (-\infty, a) \quad J_n = I_n \cap (a, b), \quad R_n = I_n \cap (b, \infty)$

Then  $\{I_n\}_{n=1}^{\infty}$  is a cover of  $E \cap (a, b)$  by open intervals

$\lambda^*(E \cap (a, b)) \leq \sum_{n=1}^{\infty} \ell(J_n)$  Also let

$$\{K_n\}_{n=1}^{\infty} = \{L_n, R_n\}_{n=1}^{\infty} \cup \left\{ \left(a - \frac{\varepsilon}{8}, a + \frac{\varepsilon}{8}\right), \left(b - \frac{\varepsilon}{8}, b + \frac{\varepsilon}{8}\right) \right\}$$

a cover of  $E \setminus (a, b)$ , so we have  $\text{length} = \varepsilon/4$

$$\lambda^*(E \setminus (a, b)) \leq \sum_{n=1}^{\infty} L(K_n) \quad ; \quad \text{we have that}$$

$$\{J_n, K_n\}_{n=1}^{\infty} \text{ covers } E$$

$$\begin{aligned} \text{so } \lambda^*(E \cap (a, b)) + \lambda^*(E \setminus (a, b)) &\leq \sum_{n=1}^{\infty} (L(J_n) + L(K_n)) \\ &= \sum_{n=1}^{\infty} L(J_n) + \sum_{n=1}^{\infty} L(K_n) < \sum_{n=1}^{\infty} L(J_n) + \sum_{n=1}^{\infty} (L(L_n) + L(R_n)) + \frac{\varepsilon}{2} \\ &= \sum_{n=1}^{\infty} (L(J_n) + L(L_n) + L(R_n)) + \frac{\varepsilon}{2} \\ &= \sum_{n=1}^{\infty} L(J_n) + \frac{\varepsilon}{2} < \lambda^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \lambda^*(E) + \varepsilon \end{aligned}$$

Since  $\varepsilon > 0$  may be chosen arbitrarily, we see

$$\lambda^*(E \cap (a, b)) + \lambda^*(E \setminus (a, b)) \leq \lambda^*(E) \quad \square$$

Corollary: If  $G \subset \mathbb{R}$  is open, then  $G \in \mathcal{L}(\mathbb{R})$

Proof: By Assign #1, we may write

$$G = \bigcup_{i=1}^{\infty} (a_i, b_i) \quad a_i < b_i \text{ in } \mathbb{R}. \quad \text{It is possible that}$$

up to two of these intervals are infinite, i.e.

$(a, \infty)$  or  $(-\infty, b)$  However, for any  $a \in \mathbb{R}$

$$(a, \infty) = \bigcup_{n=1}^{\infty} (a, n) \in \mathcal{L}(\mathbb{R})$$

recall  $\emptyset \in \mathcal{L}(\mathbb{R})$

$$\text{Similarly } (-\infty, a) = \bigcup_{n=1}^{\infty} (-n, a) \in \mathcal{L}(\mathbb{R})$$

Thus each  $(a_i, b_i) \in \mathcal{L}(\mathbb{R})$ , hence so too is  $G = \bigcup_{i=1}^{\infty} (a_i, b_i) \quad \square$

Towards Borel sets:

Definition: Let  $X$  be an arbitrary set, and  $\mathcal{P}(X) = \{E : E \subset X\}$

A family  $\mathcal{M} \subset \mathcal{P}(X)$  is called a Boolean algebra if

(i)  $\emptyset, X \in \mathcal{M}$

(ii)  $A \in \mathcal{M} \Rightarrow A^c = X \setminus A \in \mathcal{M}$

(iii)  $A_1, \dots, A_n \in \mathcal{M} \Rightarrow A_1 \cup \dots \cup A_n \in \mathcal{M}$

Moreover, we say  $\mathcal{M}$  is a  $\sigma$ -algebra if  
 (iii) For any  $A_1, A_2, \dots \in \mathcal{M}$  we have  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$

Proposition: If  $\mathcal{M} \subset \mathcal{P}(X)$  is a  $\sigma$ -algebra.

and  $A_1, A_2, \dots \in \mathcal{M}$  then  $\bigcap_{i=1}^{\infty} A_i \in \mathcal{M}$  too.

proof: (De Morgan's Law)  $\Rightarrow$

$$A_i \in \mathcal{M} \Rightarrow X \setminus A_i \in \mathcal{M} \text{ for each } i$$

Hence  $\bigcup_{i=1}^{\infty} (X \setminus A_i) = X \setminus \bigcap_{i=1}^{\infty} A_i \in \mathcal{M}$  too.

Thus  $\bigcap_{i=1}^{\infty} A_i = X \setminus (X \setminus \bigcap_{i=1}^{\infty} A_i) \in \mathcal{M}$ .  $\square$

Example:

(i) Trivial  $\sigma$ -algebra:  $\{\emptyset, X\}$  is a  $\sigma$ -algebra.

(ii)  $\mathcal{P}(X)$  is always a  $\sigma$ -algebra.

(iii) Let  $\{\mathcal{M}_\alpha\}_{\alpha \in A}$  be a collection of  $\sigma$ -algebras over index set

Then  $\mathcal{M} = \bigcap_{\alpha \in A} \mathcal{M}_\alpha$  is a  $\sigma$ -algebra.

Proof: First,  $\emptyset, X \in \mathcal{M}_\alpha$  for each  $\alpha \Rightarrow \emptyset, X \in \bigcap_{\alpha \in A} \mathcal{M}_\alpha$  too.

$B \in \mathcal{M}_\alpha$  for each  $\alpha, X \setminus B \in \mathcal{M}_\alpha$  for each  $\alpha \Rightarrow$

$$X \setminus B \in \bigcap_{\alpha \in A} \mathcal{M}_\alpha$$

$B_1, B_2, \dots \in \mathcal{M}_\alpha$  for each  $\alpha, \bigcup_{i=1}^{\infty} B_i \in \mathcal{M}_\alpha$  for each  $\alpha,$

$$\Rightarrow \bigcup_{i=1}^{\infty} B_i \in \bigcap_{\alpha \in A} \mathcal{M}_\alpha \quad \square$$

(iv)  $\mathcal{B}(\mathbb{R}) = \bigcap \{ \mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{-algebra in } \mathcal{P}(\mathbb{R}) \text{ st. } \mathcal{M} \text{ contains all open sets} \}$

called to Borel  $\sigma$ -algebra, its elements are called Borel Sets.

Plainly,  $\mathcal{B}(\mathbb{R})$  is the smallest  $\sigma$ -algebra, containing all open sets.

Corollary:  $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R})$

Proof: Collect to get two facts that all open sets are measurable, and  $\mathcal{L}(\mathbb{R})$  is itself a  $\sigma$ -algebra  $\square$

## Extended Borel sets

Notation Let  $X$  be a set  $\mathcal{A} \subset \mathcal{P}(X)$

$$\text{Let } \mathcal{A}_\sigma = \left\{ \bigcup_{i=1}^{\infty} A_i, A_1, A_2 \in \mathcal{A} \right\}$$

$$\mathcal{A}_\delta = \left\{ \bigcap_{i=1}^{\infty} A_i, A_1, A_2, \dots \in \mathcal{A} \right\}$$

Let  $\mathcal{G} = \{ G \in \mathcal{P}(\mathbb{R}) : G \text{ is open} \}$

$\mathcal{F} = \{ F \in \mathcal{P}(\mathbb{R}) : F \text{ is closed} \}$   $F \text{ closed} \Leftrightarrow \mathbb{R} \setminus F \text{ open}$

Thus we can form collections

$\mathcal{G}_\delta$  -  $\mathcal{G}_\delta$ -sets

$\mathcal{F}_\sigma$  -  $\mathcal{F}_\sigma$ -sets

Note  $\mathcal{G}$  is closed under arbitrary unions  $\Rightarrow$

$$\mathcal{G}_\sigma = \mathcal{G}$$

$\mathcal{F}$  is closed under arbitrary intersections  $\Rightarrow \mathcal{F}_\delta = \mathcal{F}$

Proposition:  $\mathcal{G} \subset \mathcal{F}_\sigma$   $\mathcal{F} \subset \mathcal{G}_\delta$

Proof: If  $G \in \mathcal{G}$ , then  $G = \bigcup_{i=1}^{\infty} (a_i, b_i)$

Let for each  $n$ ,  $F_n = \bigcup_{i=1}^{\infty} [a_i + \frac{1}{n}, b_i - \frac{1}{n}]$  ( $[c, d] = \emptyset$  if  $d < c$ )

Note, we may have an infinite interval in the decomposition of  $G$   
say, for example  $(a_1, b_1) = (a, \infty)$  In this case

$$F_n = [a_1 + \frac{1}{n}, \infty) \cup \bigcup_{i=2}^{\infty} [a_i + \frac{1}{n}, b_i - \frac{1}{n}]$$

Then each  $F_n$  is closed

and  $\bigcup_{n=1}^{\infty} F_n = G$ , thus  $G$  is an

$\mathcal{F}_\sigma$ -set.

If  $F \in \mathcal{F}$ ,  $\mathbb{R} \setminus F \in \mathcal{G}$  so  $\mathbb{R} \setminus F = \bigcup_{n=1}^{\infty} F_n$  for each sets

$F_1, F_2, \dots$  as above, thus

$$F = \mathbb{R} \setminus (\mathbb{R} \setminus F) = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} (\mathbb{R} \setminus F_n) \in \mathcal{G}_\delta \quad \square$$

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Extent of Borel set

$\mathcal{M} \subset \mathcal{P}(\mathbb{R})$

define

$$\mathcal{M}_\sigma = \left\{ \bigcup_{i=1}^{\infty} A_i, A_1, A_2, \dots \in \mathcal{M} \right\}$$

$$\mathcal{M}_\delta = \left\{ \bigcap_{i=1}^{\infty} A_i, A_1, A_2, \dots \in \mathcal{M} \right\}$$

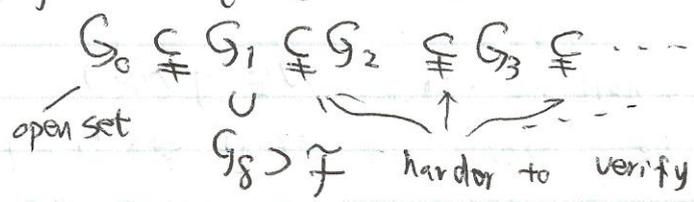
$G$  - open sets  $F$  - closed set

$G_\delta$  - sets:  $G_\delta$   $F_\sigma$  - sets:  $F_\sigma$

What do Borel sets look like?

$$G_0 = G \quad n \in \mathbb{N} \cup \{0\}, \quad G_{n+1} = ((G_n)_\delta)_\sigma$$

thus we get a sequence



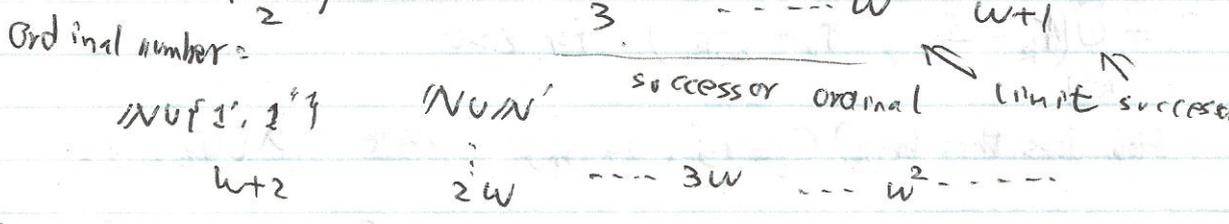
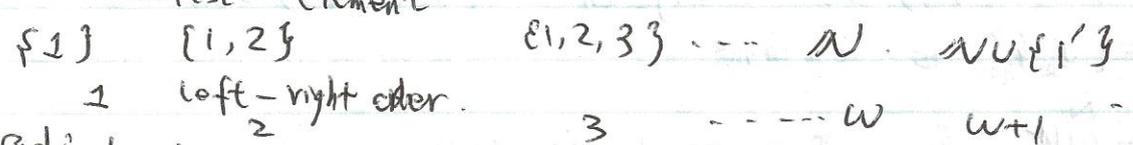
Naive hope  $B(\mathbb{R}) = \bigcup_{n=1}^{\infty} G_n$  Not true! (probably,  $\neq$ )  
 $\uparrow$   
 Borel sets

$$G_\omega = \bigcup_{n=1}^{\infty} G_n \quad ; \quad \text{sad fact } ((G_\omega)_\delta)_\sigma \subsetneq G_\omega$$

Countable ordinals: well orderings of a countable set  
 relation  $\leq$  on  $S$

- (i) total ordering  $S, t \in S$  either  $S \leq t$  or  $t \leq S$
- (ii) anti-reflective  $S \leq t, t \leq S \Rightarrow S = t$
- (iii) transitive  $S \leq t, t \leq U \Rightarrow S \leq U$

(iv) well ordering: any  $A \subset S, A \neq \emptyset$  has a smallest element



Let  $\alpha$  be a countable ordinal

$$G_{\alpha+1} = ((G_\alpha)_\delta)_\sigma \quad (\text{successor ordinal})$$

$$G_\alpha = \bigcup_{\beta < \alpha} G_\beta \quad (\text{if } \alpha \text{ itself is a limit ordinal})$$

Facts: any two ordinals are comparable.

Q1) there is a set  $[0, \aleph_1)$  of all countable ordinals.

Horrible fact.

$$\mathcal{B}(\mathbb{R}) = \bigcup_{A \in \mathcal{C}(\mathbb{R})} \mathcal{G}_A$$

Lebesgue Null sets.

Definition A subset  $N \subset \mathbb{R}$  is called a (Lebesgue) null set if

$$\lambda^*(N) = 0$$

proposition if  $N \subset \mathbb{R}$  is a null set, then  $N \in \mathcal{L}(\mathbb{R})$

PF: First observe if  $M \subset \mathbb{R}$ , then  $0 \leq \lambda^*(M) \leq \lambda^*(N)$  by increasing

$$\text{so } \lambda^*(M) = 0$$

Now, if  $E \subset \mathbb{R}$ ,

$$\lambda^*(E) \leq \lambda^*(E \cap N) + \lambda^*(E \setminus N) \leq \lambda^*(E)$$

$$\text{σ-subadditivity } = 0 \leq \lambda^*(E)$$

and thus  $N \in \mathcal{L}(\mathbb{R})$   $\square$

property:

$A_1, A_2, \dots$  are nullsets of  $\mathbb{R}$ .

Corollary if

then so too is  $\bigcup_{i=1}^{\infty} A_i$ .

proof: σ-subadditivity of  $\lambda^*$ .  $\square$

Example:  $\mathbb{Q}$  is null. Trivially,  $\{\gamma\}$

$$\lambda^*(\{\gamma\}) = 0 \quad \text{singleton} \quad (\lambda(\{\gamma\}) = 0)$$

$\mathbb{Q} = \{\gamma_k\}_{k=1}^{\infty}$  is countable.

$$G_\varepsilon = \bigcup_{k=1}^{\infty} \left( \gamma_k - \frac{\varepsilon}{2 \cdot 2^k}, \gamma_k + \frac{\varepsilon}{2 \cdot 2^k} \right) \text{ for } \varepsilon > 0$$

How does this break?  $G_\varepsilon = \bigcup_{i=1}^{\infty} (a_i, b_i)$  note  $\lambda(G_\varepsilon) \leq \varepsilon$ .

Cantor set.

$$= C_0 \setminus I_1$$

Construction  $C_0 = [0, 1]$   $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] = C_0 \setminus (I_{21} \cup I_{22})$$

$$C_n = C_{n-1} \setminus \bigcup_{k=1}^{2^{n-1}} I_{n,k} \quad (n > 1)$$

where  $I_{n,k}$  is defined as the middle third of the  $k$ -th interval of  $C_{n-1}$ .

Fact: Define the Cantor set by  $C = \bigcap_{n=1}^{\infty} C_n$

Fact:  $C \neq \emptyset$ ,

proof: Each  $C_n$  is closed,  $C_1 \supset C_2 \supset C_3 \supset \dots$   
(compact here)

By finite intersection property.

$$C = \bigcap_{n=1}^{\infty} C_n \neq \emptyset \quad \square$$

Proposition: Let  $C$  be Cantor set given above. Then

(i)  $C$  is compact hence  $C \in \mathcal{L}(\mathbb{R})$

(ii)  $\lambda(C) = 0$

proof: Assignment 2.

Proposition Let  $|C|$  denote the cardinality of  $C$ .

Then  $|C| = |\mathbb{R}| = \mathfrak{c}$

proof: Every element of  $\mathbb{R}$  "continuum" In particular,  $C$  is uncountable.

$[0, 1]$  admits a ternary expansion:

$$t = 0.\varepsilon_1\varepsilon_2\dots = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{3^k} \quad \varepsilon_k \in \{0, 1, 2\}$$

Problem: ternary expansion aren't unique.

$$0.022222 = 0.1000\dots$$

we show that

$C = \{t \in [0, 1] \mid t \text{ admits a ternary expansion with no } 1\text{'s}\}$

$$\text{Ex. } \frac{1}{3} = 0.1000\dots = 0.\underbrace{0222\dots}_{\text{no } 1\text{'s}} \Rightarrow \frac{1}{3} \in C$$

Note that we have

$$I_1 = \left(\frac{1}{3}, \frac{2}{3}\right) = \{0.\varepsilon_1\varepsilon_2\varepsilon_3 : \varepsilon_L \neq 2 \text{ for some } L > 1, \text{ and}$$

$$I_2 = \left(\frac{1}{9}, \frac{2}{9}\right) = \{0.\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 : \text{not all } \varepsilon_L\text{'s are } 0 \text{ for } L > 1$$

$$\varepsilon_L \neq 2 \text{ for some } L > 2 \text{ and}$$

$$\text{not all } \varepsilon_L\text{'s are } 0 \text{ for } L > 2\}$$

$$\vdots$$

$$I_{n,k} = \{0.\varepsilon_1\dots\varepsilon_{n-1}1\varepsilon_{n+1}\varepsilon_{n+2} : \varepsilon_L \neq 2 \text{ for some } L > n$$

$$\text{not all } \varepsilon_L\text{'s are } 0 \text{ for } L > n$$

$$k = 1 + \sum_{L=1}^{n-1} \frac{\varepsilon_L}{2^{k-1}}$$

Hilbert

$$C = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k}$$

all elements of  $[0, 1]$  which must have a 1 in

the ternary expansion.

Let  $\varphi: C \rightarrow \{0, 1\}^{\mathbb{N}}$  be given by

$$\varphi(0.\varepsilon_1\varepsilon_2\varepsilon_3\dots) = \left(\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2}, \frac{\varepsilon_3}{2}, \dots\right)$$

Check that  $\varphi$  is a bijection  $\square$

Jan 16, 2009 PMath 354 Measure Theory & Fourier Analysis

Assignment #2 - being handed out today due Jan 30

Assignment 1 solns will be posted soon

Translation invariance of Lebesgue measure

Notation: if  $E \subset \mathbb{R}$ , let  $x+E = \{x+y : y \in E\}$  denote the translate of  $E$  by  $x$ .

Proposition: Let  $E \subset \mathbb{R}$ ,  $x \in \mathbb{R}$ .

- (i)  $\lambda^*(x+E) = \lambda^*(E)$
- (ii)  $E \in \mathcal{L}(\mathbb{R}) \Rightarrow x+E \in \mathcal{L}(\mathbb{R})$
- (iii) if  $E \in \mathcal{L}(\mathbb{R})$ ,  $\lambda(x+E) = \lambda(E)$

Proof (i) We have that

$$\{I_n\}_{n=1}^{\infty} \text{ is a cover of } E \text{ by open intervals} \iff \{x+I_n\}_{n=1}^{\infty} \text{ is a cover of } x+E \text{ by open intervals.}$$

Note: if  $\{I_n\}_{n=1}^{\infty}$  is a cover of  $x+E$  by open intervals.

then  $\{-x+I_n\}_{n=1}^{\infty}$  is a cover of  $E$  by open intervals.

We have  $E$  open interval  $(a,b) \subset \mathbb{R}$ ,  $x+(a,b) = (xa, xb)$  and

$$\text{have } \ell(x+(a,b)) = (b+x) - (a+x) = b-a = \ell(a,b)$$

Use defn. of  $\lambda^*(E)$ ,  $\lambda^*(x+E)$

(ii) If  $A \in \mathcal{P}(\mathbb{R})$ , we have

$$\lambda^*(A \cap (x+E)) + \lambda^*(A \setminus (x+E))$$

$$\stackrel{\text{by (i)}}{=} \lambda^*(-x+A \cap (x+E)) + \lambda^*(-x+A \setminus (x+E))$$

$$= \lambda^*((-x+A) \cap E) + \lambda^*((-x+A) \setminus E)$$

$[E \text{ is measurable}]$

$$= \lambda^*(\mathbb{R} - X + A) = \lambda^*(A)$$

(iii) Collect (i) and (ii) we call  $\lambda = \lambda^*|_{\mathcal{L}(\mathbb{R})}$   $\square$

Question: Does there exist a non-measurable set?

Answer: Yes, accepting axiom of choice.

Theorem: Given  $a > 0$ , there exists  $E \subset (-a, a)$  s.t.

$E \notin \mathcal{L}(\mathbb{R})$  i.e.  $E$  is non-measurable.

Proof: Define an equivalence relation on  $(-a, a)$  by

$$x \sim y \iff x - y \in \mathbb{Q} \cap (-2a, 2a)$$

This is indeed an equivalence relation.

(identity)  $x \sim x$  since  $x - x = 0 \in \mathbb{Q}$

(reflexivity)  $x \sim y \Rightarrow y \sim x$  since  $x - y \in \mathbb{Q} \Rightarrow y - x = -(x - y) \in \mathbb{Q}$

(transitivity)  $x \sim y, y \sim z \Rightarrow x \sim z$  since  $x - y \in \mathbb{Q}, y - z \in \mathbb{Q}$

$$x - z = (x - y) + (y - z) \in \mathbb{Q} \quad (\mathbb{Q} \text{ is a field})$$

Let  $E \subset (-a, a)$  be such that

$$(i) \quad E \cap [x] = \emptyset$$

Notation: We define for  $x \in (-a, a)$

$$[x] = \{y \in (-a, a) : x \sim y\} = \{r \in (-a, a) : x - r \in \mathbb{Q}\} \\ = (x + \mathbb{Q}) \cap (-a, a)$$

We will choose a set  $E \subset (-a, a)$

(i)  $x, y \in E$ , and  $x \sim y$  then  $x = y$  and

(ii)  $E \cap [x] = \emptyset$  then has exactly one element for each  $x \in (-a, a)$

Note we used Axiom of choice to obtain  $E$

Let us note that  $E \cap (r + E) = \emptyset$  for  $r \in \mathbb{Q}$

Let  $\{r_k\}_{k=1}^{\infty}$  be an enumeration  $\mathbb{Q} \cap (-2a, 2a)$ , then

we have

$$(-a, a) \subset \bigcup_{k=1}^{\infty} (r_k + E) \subset (-3a, 3a) \quad (*)$$

To see the first inclusion, note that if

$$x \in (-a, a), \quad E \cap [x] = \{x_E\}$$

$\nearrow$  single point in this intersection.

We have  $x - x_E \in \mathbb{Q}$  since  $x \sim x_E$  and since

$x, x_E \in (-a, a)$  we have  $|x - x_E| < 2a$

The second inclusion in (\*) follows since

$E \subset (-a, a)$ ,  $r_k \in (-2a, 2a)$  we have for each  $x \in E$ ,

$x + r_k \in (-3a, 3a)$ .

Let us prove that  $E \notin \mathcal{L}(\mathbb{R})$

First, if  $\lambda^*(E) = 0$  so  $E$  would be a null set, hence measurable

then  $\lambda^*(r_k + E) = \lambda^*(E) = 0$  for each  $k$  and thus we would have

$$0 < 2a = \lambda(-a, a) \leq \sum_{k=1}^{\infty} \lambda^*(r_k + E) = \sum_{k=1}^{\infty} 0 = 0 \text{ which is absurd.}$$

by (\*)

Thus, if  $E$  were measurable, we would have  $\lambda(E) = \alpha > 0$

of course as  $E \subset (-a, a)$   $\alpha \leq \lambda(-a, a) = 2a < \infty$

But this by (\*) we have

$$6a = \lambda(-3a, 3a) \geq \lambda\left(\bigcup_{k=1}^n (r_k + E)\right) = \sum_{k=1}^n \lambda(r_k + E) = \sum_{k=1}^n \lambda(E) = n\alpha$$

by (\*)

for any  $n \in \mathbb{N}$ . Thus  $6a \geq n\alpha$  for any  $n \in \mathbb{N}$ . which implies that  $\alpha = 0$ .

but this is the case we just dismissed!

Conclusion  $E \notin \mathcal{L}(\mathbb{R})$   $\square$

Note (1) In assign #2, the notion of Lebesgue inner measure

is defined by:  $\lambda_*(E) = \sup \{ \lambda(K) \mid K \text{ compact, } K \subset E \}$

Note that for  $E$  as above,

if  $K \subset E$  and  $K$  is compact, we have

$$K \cap (r + K) \subset E \cap (r + E) = \emptyset \text{ if } r \in \mathbb{Q}, \text{ so}$$

as in (\*)

$$\bigcup_{k=1}^n (r_k + K) \subset \bigcup_{k=1}^n (r_k + E) \subset (-3a, 3a),$$

since each  $r_k + K$  is measurable for any  $n \in \mathbb{N}$

$$\sum_{k=1}^{\infty} \lambda(r_k + K) = \lambda\left(\bigcup_{k=1}^{\infty} (r_k + K)\right) \leq \lambda(-3a, 3a) = 6a$$

$$\Rightarrow n \lambda(K) \leq 6a \Rightarrow \lambda(K) = 0$$

so conclusion  $\lambda_x(E) = 0$

(23) if we negate axiom of choice, even replacing by "countable choice", then every subset  $E \subset \mathbb{R}$  is Lebesgue measurable. R. M. Solovay Ann of Math (2), v.92, 1970

Jan 19, 2009 PMath 354 Measure Theory and Fourier Analysis

Towards the definition of the Lebesgue integral

Measurable functions:

Notation:  $f: \mathbb{R} \rightarrow \mathbb{R}$   $A \subset \mathbb{R}$   $f^{-1}(A) = \{x \in \mathbb{R}; f(x) \in A\}$

Definition: Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  we set

Say  $f$  is measurable, if

$$f^{-1}((\alpha, \infty)) = \{x \in \mathbb{R}; f(x) > \alpha\} \in \mathcal{L}(\mathbb{R}) \text{ for every } \alpha \in \mathbb{R}$$

Examples: (i) if  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous, then  $f$  is measurable.

Proof:  $(\alpha, \infty)$  open  $\Rightarrow f^{-1}((\alpha, \infty))$  open for any  $\alpha \in \mathbb{R}$ .

[property of continuity]

(ii) if  $A \subset \mathbb{R}$ , define its indicator / characteristic function

$$\chi_A: \mathbb{R} \rightarrow \mathbb{R}$$

↳ chi

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}$$

Then  $\chi_A$  is measurable  $\Leftrightarrow A \in \mathcal{L}(\mathbb{R})$

proof: let  $\alpha \in \mathbb{R}$ .

$$\chi_A^{-1}((\alpha, \infty)) = \begin{cases} \emptyset & \alpha \geq 1 \\ A & 0 \leq \alpha < 1 \\ \mathbb{R} & \alpha < 0 \end{cases}$$

since  $\emptyset, \mathbb{R} \in \mathcal{L}(\mathbb{R})$  we find

$$\chi_A \text{ measurable} \Leftrightarrow A \in \mathcal{L}(\mathbb{R}) \quad \square$$

Proposition: Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $\alpha \in \mathbb{R}$ , then TFAE.

(i)  $f$  is measurable i.e.  $f^{-1}((\alpha, \infty)) \in \mathcal{L}(\mathbb{R})$  for each  $\alpha \in \mathbb{R}$

(i')  $f^{-1}((-\infty, \alpha]) \in \mathcal{L}(\mathbb{R})$  for  $\alpha \in \mathbb{R}$

(i'')  $f^{-1}((-\infty, \alpha)) \in \mathcal{L}(\mathbb{R})$  for  $\alpha \in \mathbb{R}$ .

proof: (i)  $\Leftrightarrow$  (i')

$$f^{-1}((-\infty, \alpha]) = \{x \in \mathbb{R}; f(x) \leq \alpha\}$$

$$= \mathbb{R} \setminus \{x \in \mathbb{R}; f(x) > \alpha\} = \mathbb{R} \setminus f^{-1}((\alpha, \infty))$$

for all  $\alpha \in \mathbb{R}$ . Hence  $f^{-1}((-\infty, \alpha]) \in \mathcal{L}(\mathbb{R}) \Leftrightarrow f^{-1}((\alpha, \infty)) \in \mathcal{L}(\mathbb{R})$

(i')  $\Rightarrow$  (i'') Note  $(-\infty, \alpha) = \bigcup_{n=1}^{\infty} (-\infty, \alpha - \frac{1}{n}]$

Hilroy

and hence  $f^{-1}((-\infty, \alpha)) = \bigcup_{n=1}^{\infty} f^{-1}((-\infty, \alpha - \frac{1}{n}])$

so  $f^{-1}((-\infty, \alpha)) \in \mathcal{L}(\mathbb{R})$  provided each  $f^{-1}((-\infty, \alpha - \frac{1}{n}]) \in \mathcal{L}(\mathbb{R})$

(iii)  $\Rightarrow$  (i) As before, for each  $\alpha \in \mathbb{R}$ ,

$f^{-1}([\alpha, \infty)) \in \mathcal{L}(\mathbb{R})$  given condition (iii)

By same argument (ii)  $\Rightarrow$  (iii) we obtain (i)  $\square$

Proposition: Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ , then

$f$  is measurable  $\Leftrightarrow f^{-1}(A) \in \mathcal{L}(\mathbb{R})$  for each  $A \in \mathcal{B}(\mathbb{R})$

proof: " $\Leftarrow$ " is clear.

$\mathcal{B}(\mathbb{R})$   $\uparrow$  sigma algebra

" $\Rightarrow$ " will be done in stages.

open intervals  $a < b$  in  $\mathbb{R}$ .  $(a, b) = (a, \infty) \cap (-\infty, b)$

$$f^{-1}((a, b)) = f^{-1}((a, \infty)) \cap f^{-1}((-\infty, b))$$

From above,  $f^{-1}((a, \infty)), f^{-1}((-\infty, b)) \in \mathcal{L}(\mathbb{R})$  so

$$f^{-1}((a, b)) \in \mathcal{L}(\mathbb{R}).$$

open sets  $G = \bigcup_{i=1}^{\infty} (a_i, b_i)$

$$f^{-1}(G) = \bigcup_{i=1}^{\infty} f^{-1}((a_i, b_i)) \in \mathcal{L}(\mathbb{R})$$

since each  $f^{-1}((a_i, b_i)) \in \mathcal{L}(\mathbb{R})$  from above.

Borel sets: Let  $\mathcal{M}_f = \{A \subset \mathbb{R} : f^{-1}(A) \in \mathcal{L}(\mathbb{R})\}$

From above,  $\mathcal{M}_f$  contains all open sets.

Let us see that  $\mathcal{M}_f$  is a  $\sigma$ -algebra of sets

(i)  $\emptyset, \mathbb{R} \in \mathcal{M}_f$  since  $f^{-1}(\emptyset) = \emptyset \in \mathcal{L}(\mathbb{R})$

(open sets)

$$f^{-1}(\mathbb{R}) = \mathbb{R} \in \mathcal{L}(\mathbb{R})$$

(ii)  $A \in \mathcal{M}_f \Rightarrow \mathbb{R} \setminus A \in \mathcal{M}_f$  since  $f^{-1}(\mathbb{R} \setminus A) = \mathbb{R} \setminus f^{-1}(A)$

(iii) if  $A_1, A_2, \dots \in \mathcal{M}_f \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}_f$ , since

$$f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$$

We have  $\mathcal{M}_f$  is a  $\sigma$ -algebra which contains all open sets.

Thus  $\mathcal{M}_f \supset \mathcal{B}(\mathbb{R})$  by definition of Borel  $\sigma$ -algebra  $\square$

Proposition (Algebraic properties of measurable functions)

Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be measurable,  $c \in \mathbb{Q}$ , and  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  be continuous, then the following functions are measurable.

- (i)  $cf$  [ $cf(x) = c[f(x)]$ ]
- (ii)  $f+g$  [ $(f+g)(x) = f(x)+g(x)$ ]
- (iii)  $\varphi \circ f$  [ $(\varphi \circ f)(x) = \varphi(f(x))$ ]
- (iv)  $fg$  [ $fg(x) = f(x) \cdot g(x)$ ]

Note that Let

$$\mathcal{M}(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is measurable}\}$$

Properties (i) (ii) and (iv) tell us the  $\mathcal{M}(\mathbb{R})$  is an algebra of functions.

(iii) tells us that  $\mathcal{M}(\mathbb{R})$  has a "continuous functional calculus"

Proof: (i)  $\alpha \in \mathbb{R}$ , we have

$$(cf)^{-1}((\alpha, \infty)) = \begin{cases} f^{-1}((\frac{\alpha}{c}, \infty)) & c > 0 \\ \emptyset & c = 0 \quad \alpha \leq 0 \\ \emptyset & c = 0 \quad \alpha > 0 \\ f^{-1}((-\infty, \frac{\alpha}{c})) & c < 0 \end{cases}$$

and this set is measurable if  $f$  is measurable

(ii) Let  $\mathcal{Q} = \{\sigma_k\}_{k=1}^{\infty}$  if  $\alpha \in \mathbb{R}$ , we have

$$\begin{aligned} (f+g)^{-1}((\alpha, \infty)) &= \{x \in \mathbb{R} : f(x) + g(x) > \alpha\} \\ &= \{x \in \mathbb{R} : f(x) > \alpha - g(x)\} \\ &= \bigcup_{k=1}^{\infty} (\{x \in \mathbb{R} : f(x) > \sigma_k\} \cap \{x \in \mathbb{R} : \sigma_k > \alpha - g(x)\}) \\ &= \bigcup_{k=1}^{\infty} (f^{-1}((\sigma_k, \infty)) \cap g^{-1}((\alpha - \sigma_k, \infty))) \end{aligned}$$

(iii) Let  $\alpha \in \mathbb{R}$ . since each  $f^{-1}((\sigma_k, \infty)), g^{-1}((\alpha - \sigma_k, \infty)) \in \mathcal{M}(\mathbb{R})$

$$(\varphi \circ f)^{-1}((\alpha, \infty)) = f^{-1}(\underbrace{\varphi^{-1}((\alpha, \infty))}_{\text{open}}) \in \mathcal{M}(\mathbb{R})$$

(iv) Let us observe that  $\varphi$  is continuous  $\Rightarrow \varphi^{-1}((\alpha, \infty))$  open

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$$

where each of  $f+g, f-g$  is measurable by (ii)

and  $-g$  measurable by (i),

we use  $\phi(x) = x^2$  is continuous. use part (iii) then (ii)  
 then (i), to finish  $\square$

Corollary: If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is measurable, then so too are

$$|f|, f^+, f^-$$

where  $|f|(x) = |f(x)|$

$$f^+(x) = \max\{f(x), 0\} \quad f^-(x) = \max\{-f(x), 0\}$$

$$|f| = f^+ - f^-$$

proof  $|f|$  is measurable by (iii)

$$f^+ = \frac{1}{2}(|f| + f) \quad f^- = \frac{1}{2}(|f| - f) \quad \square$$

Jan 21. Wed. 2009 PMath 354

ERRATUM from last class.

If  $f \in \mathcal{M}(\mathbb{R})$  (measurable functions from  $\mathbb{R} \rightarrow \mathbb{R}$ )

$c, \alpha \in \mathbb{R}$  then

$$(cf)^+((\alpha, \infty)) = \begin{cases} f^+((\frac{\alpha}{c}, \infty)) & \text{if } c > 0 \\ \mathbb{R} \text{ [or } \emptyset] & \text{if } c = 0 \text{ } \alpha < 0 \\ \emptyset & \text{if } c = 0 \text{ } \alpha \geq 0 \end{cases}$$

$$f^+((-\infty, \frac{\alpha}{c}))$$

we saw,  $\mathcal{M}(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ measurable}\}$

then this is an algebra of functions.

$$f+g, cf, fg$$

Goal: Limits of sequences of elements from  $\mathcal{M}(\mathbb{R})$

Formalism to deal with limits of functions

Extended real numbers  $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$

If  $f: \mathbb{R} \rightarrow \bar{\mathbb{R}}$ , we call  $f$  "extended real-valued".

We say  $f: \mathbb{R} \rightarrow \bar{\mathbb{R}}$  is measurable if  $f^{-1}((\alpha, \infty]) \in \mathcal{L}(\mathbb{R})$  for  $\alpha \in \mathbb{R}$

Note: By similar methods as we used before we have

$$f: \mathbb{R} \rightarrow \bar{\mathbb{R}} \text{ is measurable} \iff f^{-1}(A) \in \mathcal{L}(\mathbb{R}) \text{ for each } A \in \mathcal{P}(\bar{\mathbb{R}})$$

$$\& f^{-1}(\{\infty\}), f^{-1}(\{-\infty\}) \in \mathcal{L}(\mathbb{R})$$

Example:  $f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} f^{-1}((n, \infty])$  (check)

Let  $\bar{\mathcal{M}}(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \bar{\mathbb{R}} \mid f \text{ is measurable}\}$  21.

Warning (i) if  $f, g \in \bar{\mathcal{M}}(\mathbb{R})$  we may not be able to make sense of  $f+g$ . Say  $f(x_0) = \infty, g(x_0) = -\infty$

There is no meaningful way to assign meaning to  $(f+g)(x_0)$   
we have no reasonable

(ii) Note that if  $f \in \bar{\mathcal{M}}(\mathbb{R}), \varphi: \mathbb{R} \rightarrow \mathbb{R}$  is continuous, we may not be able to make sense of  $\varphi(f(x))$

Problem How do we assign  $\varphi(\infty)$  or  $\varphi(-\infty)$   
However, if  $\lim_{t \rightarrow \infty} \varphi(t)$  exists [ $\pm \infty$  are acceptable] and

$\lim_{t \rightarrow -\infty} \varphi(t)$  exists [ $\pm \infty$  are acceptable]

then  $\varphi \circ f \in \bar{\mathcal{M}}(\mathbb{R})$  provided we accept the convention

$$\varphi(\infty) = \lim_{t \rightarrow \infty} \varphi(t), \quad \varphi(-\infty) = \lim_{t \rightarrow -\infty} \varphi(t)$$

Example  $t \mapsto |t|, \lim_{t \rightarrow \infty} |t| = \infty$ . So we let  $|\infty| = \infty, |-\infty| = \infty$

consequence:  $f \in \bar{\mathcal{M}}(\mathbb{R}) \Rightarrow |f| \in \bar{\mathcal{M}}(\mathbb{R})$  too.

proposition: Let  $(f_n)_{n=1}^{\infty} \subset \mathcal{M}(\mathbb{R})$  [or in  $\bar{\mathcal{M}}(\mathbb{R})$ ], then the following functions are all measurable.

(i)  $\sup_{n \in \mathbb{N}} f_n$  [ $(\sup_{n \in \mathbb{N}} f_n)(x) \stackrel{\Delta}{=} \sup_{n \in \mathbb{N}} f_n(x)$ , note  $\infty$  is possible]

(ii)  $\inf_{n \in \mathbb{N}} f_n$  [ $-\infty$  is possible at some points]

(iii)  $\limsup_{n \rightarrow \infty} f_n := \lim_{n \rightarrow \infty} (\sup_{k \geq n} f_k)$

(iv)  $\liminf_{n \rightarrow \infty} f_n := \lim_{n \rightarrow \infty} (\inf_{k \geq n} f_k)$

Proof: (i)  $(\sup_{n \in \mathbb{N}} f_n)^{-1}([-\infty, a]) = \{x \in \mathbb{R} : \sup_{n \in \mathbb{N}} f_n(x) \leq a\}$   
 $= \bigcap_{n=1}^{\infty} \{x : f_n(x) \leq a\} = \bigcap_{n=1}^{\infty} f_n^{-1}([-\infty, a]) \in \mathcal{L}(\mathbb{R})$

(ii)  $(\inf_{n \in \mathbb{N}} f_n) = -(\sup_{n \in \mathbb{N}} (-f_n))$

(iii) Let  $g_n = \sup_{k \geq n} f_k$  and  $g_n \in \bar{\mathcal{M}}(\mathbb{R})$  by (i)

Also  $g_1 \geq g_2 \geq \dots$  so  $\limsup_{n \rightarrow \infty} f_n = \inf_{n \in \mathbb{N}} \sup_{k \geq n} f_k$   
 $= \inf_{n \in \mathbb{N}} g_n \in \bar{\mathcal{M}}(\mathbb{R})$  by (ii)

$$(iv) \liminf_{n \rightarrow \infty} f_n = -(\limsup_{n \rightarrow \infty} (-f_n)) \quad \square$$

Corollary: If  $(f_n)_{n=1}^{\infty} \subset \mathcal{M}(\mathbb{R})$  (or  $\mathcal{M}(\mathbb{R})$ ) and

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \text{ exists for each } x \in \mathbb{R} \text{ then } f \in \mathcal{M}(\mathbb{R})$$

[ $\pm \infty$  acceptable]

proof: If  $f = \lim_{n \rightarrow \infty} f_n$  (pointwise) exists. then  $f = \lim_{n \rightarrow \infty} \sup f_n = \lim_{n \rightarrow \infty} \inf f_n \quad \square$

Towards the Lebesgue integral simple functions.

Definition: Let  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ . We say that  $\varphi$  is simple if

$$\varphi(\mathbb{R}) = \{\varphi(x) \mid x \in \mathbb{R}\} \text{ is finite}$$

if  $A \subset \mathbb{R}$ , then  $\varphi: A \rightarrow \mathbb{R}$  is simple (on  $A$ ) if  $\varphi(A) = \{\varphi(x) \mid x \in A\}$  is finite.

Note: If  $\varphi(A)$  is finite, then we can write

$$\varphi(A) = \{\alpha_1 < \alpha_2 < \dots < \alpha_n\} \quad (\alpha_i \text{'s are distinct from one another})$$

$$\text{Let for } i=1, \dots, n \quad E_i = \varphi^{-1}(\{\alpha_i\}) = \{x \in A : \varphi(x) = \alpha_i\}$$

Then we can write

$$\varphi = \sum_{i=1}^n \alpha_i \chi_{E_i} \quad \chi_{E_i} = \begin{cases} 1 & x \in E_i \\ 0 & x \notin E_i \end{cases}$$

Also, if  $i \neq j$ ,  $E_i \cap E_j = \emptyset$ .

Proposition: Let  $A \in \mathcal{L}(\mathbb{R})$  [usually  $A = \mathbb{R}$ , interval] and  $\varphi: A \rightarrow \mathbb{R}$

is simple with  $\varphi(A) = \{\alpha_1 < \dots < \alpha_n\}$ , then

$\varphi$  is a measurable function  $\Leftrightarrow E_i = \varphi^{-1}(\{\alpha_i\})$  is measurable (as a set) (as a function)

proof: " $\Rightarrow$ " Each  $\{\alpha_i\}$  is closed, hence Borel, so

$$E_i = \varphi^{-1}(\{\alpha_i\}) \in \mathcal{L}(\mathbb{R}) \text{ by an earlier proposition.}$$

" $\Leftarrow$ " We saw earlier, when we introduced measurable functions.

that  $E_i \in \mathcal{L}(\mathbb{R}) \Leftrightarrow \chi_{E_i} \in \mathcal{M}(\mathbb{R})$ , we also have that

$\mathcal{M}(\mathbb{R})$  is closed under scalar multiplication and pointwise addition  $\square$

Let  $S(A) = \{\varphi: A \rightarrow \mathbb{R} \mid \varphi \text{ is simple and measurable}\}$

$$S^+(A) = \{\varphi \in S(A) : \varphi(x) \geq 0 \text{ for all } x \in A\}$$

Definition (pre-Lebesgue integral) If  $\varphi \in S^+(A)$  [ $A$  is measurable]

$$\text{with } \varphi(A) = \{\alpha_1 < \dots < \alpha_n\} \quad E_i = \varphi^{-1}(\{\alpha_i\})$$

then we define

$$I_A(\varphi) = \sum_{i=1}^n \alpha_i \lambda(E_i) \quad [\text{value may be } \infty]$$

for this, we accept the convention

$$\alpha \lambda(E) = \infty \quad \text{if } \alpha > 0 \text{ and } \lambda(E) = \infty$$

$$\text{and } 0 \cdot \lambda(E) = 0 \quad \text{even if } \lambda(E) = \infty$$

Jan 23, 2009 PMath 354 Measure Theory and Fourier Analysis

Last class:  $A \in \mathcal{L}(\mathbb{R})$  (often  $A = [a, b], \mathbb{R}$ )

$\varphi: A \rightarrow \mathbb{R}$  is simple if  $\varphi(A) = \{\alpha_1 < \dots < \alpha_n\}$  is finite

$$\text{Let } E_i = \varphi^{-1}(\{\alpha_i\}), \quad \varphi = \sum_{i=1}^n \alpha_i \chi_{E_i} \quad (*)$$

$\varphi$  measurable  $\Leftrightarrow$  Each  $E_i$  - meas.

$$S^+(A) = \{ \varphi: A \rightarrow \mathbb{R} \mid \varphi \text{ is simple, measurable, } \varphi(x) \geq 0, \text{ for } x \in A \}$$

"proto-integral" if  $\varphi \in S^+(A)$  as in (\*)

$$I_A(\varphi) = \sum_{i=1}^n \alpha_i \lambda(E_i) \quad (\text{may be } \infty)$$

Proposition Let  $A \in \mathcal{L}(\mathbb{R}), \varphi, \psi \in S^+(A), c \geq 0$ , then

(i)  $c\varphi \in S^+(A)$  and  $I_A(c\varphi) = c I_A(\varphi)$

(ii)  $\varphi + \psi \in S^+(A)$  and  $I_A(\varphi + \psi) = I_A(\varphi) + I_A(\psi)$

(iii) If  $\varphi \leq \psi$  (pointwise) then  $I_A(\varphi) \leq I_A(\psi)$

proof: (i) Obvious

(ii) Let  $\varphi = \{\alpha_1 < \dots < \alpha_n\}$   $E_i = \varphi^{-1}(\{\alpha_i\})$ ,  $\psi(A) = \{\beta_1 < \dots < \beta_m\}$

$F_j = \psi^{-1}(\{\beta_j\})$ , so

$$\varphi = \sum_{i=1}^n \alpha_i \chi_{E_i} \quad \psi = \sum_{j=1}^m \beta_j \chi_{F_j} \quad E_i \cap F_j = \emptyset = F_i \cap F_j \text{ for } i \neq j$$

Let  $\{\gamma_1 < \dots < \gamma_L\} = \{\alpha_i + \beta_j \mid i=1, \dots, n, j=1, \dots, m\}$   $h=1, \dots, L$  let

$$D_k = \cup_{i,j: \alpha_i + \beta_j = \gamma_k} E_i \cap F_j$$

$$\text{Then } \varphi + \psi = \sum_{i=1}^n \alpha_i \chi_{E_i} + \sum_{j=1}^m \beta_j \chi_{F_j}$$

$$= \sum_{i=1}^n \alpha_i \sum_{j=1}^m \chi_{E_i \cap F_j} + \sum_{j=1}^m \beta_j \sum_{i=1}^n \chi_{E_i \cap F_j} \quad (\text{change order of finite sums})$$

$$= \sum_{i=1}^n \sum_{j=1}^m (\alpha_i + \beta_j) \chi_{E_i \cap F_j}$$

$$= \sum_{k=1}^L \gamma_k \sum_{\{i,j\} | \alpha_i + \beta_j = \gamma_k} \chi_{E_i \cap F_j}$$

$$= \sum_{k=1}^L \gamma_k \chi_{D_k} \quad [\text{Note some } D_k \text{'s may be } \emptyset, \text{ in which case } \lambda(D_k) = \lambda(\emptyset) = 0]$$

Now we have

$$\begin{aligned} I_A(\varphi + \psi) &= \sum_{k=1}^L \gamma_k \lambda(D_k) \stackrel{(H)}{=} \sum_{k=1}^L \gamma_k \sum_{\{i,j\}: \alpha_i + \beta_j = \gamma_k} \lambda(E_i \cap F_j) \\ &= \sum_{i=1}^n \sum_{j=1}^m (\alpha_i + \beta_j) \lambda(E_i \cap F_j) \\ &= \sum_{i=1}^n \alpha_i \sum_{j=1}^m \lambda(E_i \cap F_j) + \sum_{j=1}^m \beta_j \sum_{i=1}^n \lambda(E_i \cap F_j) \\ &\stackrel{(H)}{=} \sum_{i=1}^n \alpha_i \lambda(E_i) + \sum_{j=1}^m \beta_j \lambda(F_j) \end{aligned}$$

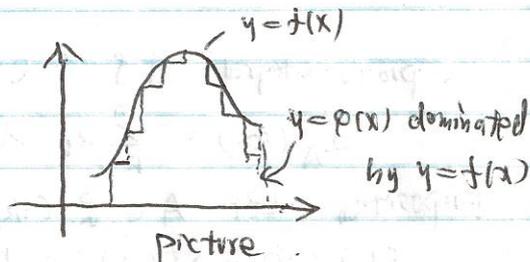
At (H) we used additivity of  $\lambda$

(iii) As in (ii) we write

$$\varphi = \sum_{i=1}^n \alpha_i \chi_{E_i} \quad \psi = \sum_{j=1}^m \beta_j \chi_{F_j}$$

since  $\varphi \leq \psi$  if  $E_i \cap F_j \neq \emptyset$

we have  $\alpha_i \leq \beta_j$  Then we have



$$\begin{aligned} I_A(\varphi) &= \sum_{i=1}^n \alpha_i \lambda(E_i) = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \lambda(E_i \cap F_j) \\ &\leq \sum_{i=1}^n \sum_{j=1}^m \beta_j \lambda(E_i \cap F_j) = \sum_{j=1}^m \beta_j \lambda(F_j) = I_A(\psi) \quad \square \end{aligned}$$

Definition = Let  $f \in \bar{M}^+(A) = \{f: A \rightarrow \bar{\mathbb{R}} = [-\infty, \infty] \mid f \text{ is measurable, } f(x) \geq 0\}$   
Fix  $A \in \mathcal{I}(\mathbb{R})$

Let  $S_f^+(A) = \{\varphi \in S^+(A) : \varphi \leq f \text{ (pointwise)}\}$

We define the Lebesgue integral of  $f$  on  $A$

$$\text{by } \int_A f = \sup \{ I_A(\varphi) : \varphi \in S_f^+(A) \} \quad [\infty \text{ is acceptable}]$$

if  $\int_A f < \infty$ , we say that  $f$  is Lebesgue integrable.

proposition (i) if  $f, g \in \bar{M}^+(A)$   $f \leq g$  on  $A$ , then  $\int_A f \leq \int_A g$

(ii) if  $f \in \bar{M}^+(A)$ ,  $B \in \mathcal{I}(\mathbb{R})$  with  $B \subset A$ , then  $\int_B f \leq \int_A f$ .

(iii) if  $\varphi \in S^+(A)$   $\int_A \varphi = I_A(\varphi)$

pf: (i)  $f \leq g$  on  $A \Rightarrow S_f^+(A) \subset S_g^+(A)$  i.e.  $\varphi \leq f \Rightarrow \varphi \leq g$  too.

$$\int_A f = \sup_{\phi \in S_f^+(A)} \int_A \phi \leq \sup_{\phi \in S_g^+(A)} \int_A \phi$$

larger set  $S_g^+(A)$

(ii) if  $\phi \in S_f^+(B)$  define  $\tilde{\phi} \in S_f^+(A)$  by  $\tilde{\phi}(x) = \begin{cases} \phi(x) & x \in B \\ 0 & x \in A \setminus B \end{cases}$

It is clear that  $\int_A \tilde{\phi} = \int_B \phi$   $\{ \tilde{\phi} = \phi \in S_f^+(B) \} \subset S_f^+(A)$

$$\int_B f = \sup_{\phi \in S_f^+(B)} \int_B \phi = \sup_{\phi \in S_f^+(B)} \int_A \tilde{\phi} \leq \sup_{\phi \in S_f^+(A)} \int_A \phi = \int_A f$$

(iii) if  $\psi \in S_\phi^+(A)$  then  $\int_A \psi \leq \int_A \phi$  by the previous Proposition (part iii) thus

$$\int_A \phi = \sup_{\psi \in S_\phi^+(A)} \int_A \psi \leq \int_A \phi$$

However  $\phi \in S_\phi^+(A)$  so  $\int_A \phi \leq \int_A \phi$ , Hence  $\int_A \phi = \int_A \phi$   $\square$   
 Note due to (iii) we will no longer use notation  $\int_A \phi$

Monotone Convergence Theorem:

if  $f_n \in L^+(A)$   $\{f_n\}_{n=1}^\infty \subset \bar{M}^+(A)$  is increasing i.e.  $f_1 \leq f_2 \leq \dots$  always  $\int_A \phi$  instead.

and  $f = \lim_{n \rightarrow \infty} f_n$  (pointwise) then

$$\int_A f = \lim_{n \rightarrow \infty} \int_A f_n$$

in particular, if  $\sup_{n \in \mathbb{N}} \int_A f_n < \infty$ , then  $f$  is Lebesgue integrable.

Example: MCT fails for Riemann integration.

$$[0,1] \cap \mathbb{Q} = \{r_k\}_{k=1}^\infty$$

Define  $f_n = \chi_{\{r_1, \dots, r_n\}}$  Then  $f_1 \leq f_2 \leq \dots$  and  $\lim_{n \rightarrow \infty} f_n = \chi_{\mathbb{Q} \cap [0,1]}$

For Riemann integration

$\int_0^1 f_n(x) dx = 0$  while  $\chi_{\mathbb{Q} \cap [0,1]}$  is NOT Riemann integrable on  $[0,1]$ .

Jan 26, 2009 PMath 354 Measure Theory and Fourier Analysis.

Assignment #1 should be graded by Wed.

Last time:  $f \in \bar{M}^+(A)$ ,  $S_f^+ = \{ \phi : A \rightarrow [0, +\infty] \mid \phi \leq f \text{ (pointwise)} \}$   $\phi$  simple, measurable

Def'n of Lebesgue integral  $\int_A f = \sup_{\mathcal{P} \in \mathcal{S}_f^+(A)} I_A(\mathcal{P})$

Monotone Convergence Theorem

Monotone

if  $(f_n)_{n=1}^\infty \subset \bar{\mathcal{M}}^+(A)$   $A \in \mathcal{L}(\mathbb{R})$ ,  $f_1 \leq f_2 \leq \dots$

and  $f = \lim_{n \rightarrow \infty} f_n$  (pointwise), then  $f \in \bar{\mathcal{M}}^+(A)$

$$\lim_{n \rightarrow \infty} \int_A f_n = \int_A f \quad [\infty \text{ is possible}]$$

Thus if  $\lim_{n \rightarrow \infty} \int_A f_n < \infty$ ,  $f$  is Lebesgue integrable.

Lemma: if  $A_1, A_2, \dots \in \mathcal{L}(\mathbb{R})$   $A_1 \subset A_2 \subset \dots$  then

$$\lambda\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n \rightarrow \infty} \lambda(A_n) \quad [\infty \text{ may be possible}]$$

Proof: Let  $C_1 = A_1$ ,  $C_2 = A_2 \setminus A_1$ ,  $C_3 = A_3 \setminus A_2$ ,  $\dots$

Then each  $C_n \in \mathcal{L}(\mathbb{R})$  and since  $A_1 \subset A_2 \subset \dots$

$$C_n \cap C_m = \emptyset \quad \text{for } n \neq m.$$

Also  $A_n = \bigcup_{k=1}^n C_k$  Thus we have

$$\lambda(A_n) = \lambda\left(\bigcup_{k=1}^n C_k\right) = \sum_{k=1}^n \lambda(C_k) \xrightarrow{n \rightarrow \infty} \sum_{k=1}^\infty \lambda(C_k) = \lambda\left(\bigcup_{k=1}^\infty C_k\right) = \lambda\left(\bigcup_{n=1}^\infty A_n\right) \quad \square$$

Proof of M.C.T.

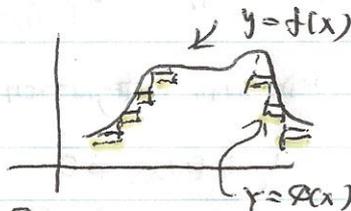
since  $f_1 \leq f_2 \leq \dots$  we have  $\int_A f_1 \leq \int_A f_2 \leq \dots$

so that  $\lim_{n \rightarrow \infty} \int_A f_n = \sup_{n \in \mathbb{N}} \int_A f_n$  and hence exists [or is  $\infty$ ]

Also. we have  $f = \lim_{n \rightarrow \infty} f_n = \sup_{n \in \mathbb{N}} f_n \in \bar{\mathcal{M}}^+(A)$

Fix. for now,  $\mathcal{P} \in \mathcal{S}_f^+(A)$ , and  $0 < \eta < 1$  and we will show that

$$(f) \quad \lim_{n \rightarrow \infty} \int_A f_n \geq \eta \int_A \mathcal{P} \quad \text{picture}$$



We prove (f) define  $A_n = \{x \in A : f_n(x) \geq \eta \mathcal{P}(x)\}$

We then have that

(i)  $A_1 \subset A_2 \subset A_3 \subset \dots$  since  $f_1 \leq f_2 \leq f_3 \leq \dots$

(ii)  $\bigcup_{n=1}^\infty A_n = A$  since  $\eta \mathcal{P}(x) < \mathcal{P}(x) \leq f(x)$   $x \in A$  and  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$

Now we let  $\varphi(A) = \{ \alpha_1 < \dots < \alpha_m \}$   $E_i = \varphi^{-1}(\{ \alpha_i \}) \subset A$   
 We then have for each  $n$ .  $\varphi = \sum_{i=1}^m \alpha_i \chi_{E_i \cap A_n}$

$$\int_A f_n \geq \int_{A_n} f_n \geq \int_{A_n} \varphi = \sum_{i=1}^m \alpha_i \lambda(E_i \cap A_n)$$

prop. Last class

$$\xrightarrow[n \rightarrow \infty]{\text{by lemma}} \sum_{i=1}^m \alpha_i \lambda(E_i) \quad A_1 \cap E_i \subset A_2 \cap E_i \subset \dots$$

$$= \int_A \varphi = \int_A \varphi$$

$\bigcup_{n=1}^{\infty} (A_n \cap E_i) = E_i \cap A = E_i$

Thus we proved (f) since (f) holds for any  $0 < \eta < 1$   
 we see that

$$\lim_{n \rightarrow \infty} \int_A f_n \geq \lim_{\eta \uparrow 1} \eta \int_A \varphi = \int_A \varphi$$

Since  $\int_A f = \sup_{\varphi \in S_f^+(A)} \int_A \varphi$  we see that

$$\lim_{n \rightarrow \infty} \int_A f_n \geq \int_A f$$

To see the converse inequality, we have  $f_n \leq f$  since

$$f = \lim_{n \rightarrow \infty} f_n = \sup_{n \in \mathbb{N}} f_n$$

so it follows  $\int_A f_n \leq \int_A f$  and hence

$$\lim_{n \rightarrow \infty} \int_A f_n \leq \int_A f \quad \square$$

Lemma If  $f: A \rightarrow [0, +\infty]$  where  $A \in \mathcal{L}(\mathbb{R})$ , then  
 $f \in \bar{M}^+(A) \iff$  there is a sequence  $(\varphi_n)_{n=1}^{\infty} \subset S^+(A)$  such that  
 $\lim_{n \rightarrow \infty} \varphi_n = f$  (pointwise)

Moreover, we can arrange  $\varphi_1 \leq \varphi_2 \leq \dots \leq f$

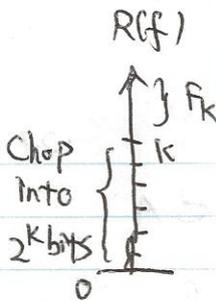
prof: " $\Leftarrow$ " "Any limit of a sequence of non-negative measurable functions, is itself a measurable (extended) non-negative-valued function."

" $\Rightarrow$ " Let for  $k \in \mathbb{N}$ .

$$F_k = \{ x \in A : f(x) \geq k \} = f^{-1}([k, \infty])$$

and for each  $i=1, 2, \dots, k \cdot 2^k$ , let

$$E_{k,i} = \{ x \in A : \frac{i-1}{2^k} \leq f(x) < \frac{i}{2^k} \} = f^{-1}([\frac{i-1}{2^k}, \frac{i}{2^k}))$$



Then  $F_k \cup \bigcup_{i=1}^{k-1} E_{k,i} = A$  Thus if we let

$$\varphi_k = k \chi_{F_k} + \sum_{i=1}^{k-1} \frac{i-1}{2^k} \chi_{E_{k,i}}$$

we have  $\varphi_k \leq f$  and  $\varphi_1 \leq \varphi_2 \leq \dots$

Since the  $E_{k+1,i}$ 's "Refine" the  $E_{k,i}$ 's □

Corollary (to Lemma and MCT)

(i) If  $f, g \in \bar{M}^+(A)$ , and  $c \geq 0$ , then

$$\int_A cf = c \int_A f \quad \text{and} \quad \int_A f+g = \int_A f + \int_A g$$

(ii) If  $(f_n)_{n=1}^{\infty} \subset \bar{M}^+(A)$  and  $f = \sum_{n=1}^{\infty} f_n$  (pointwise), then

$$\int_A f = \sum_{n=1}^{\infty} \int_A f_n$$

(iii) If  $A = \bigcup_{n=1}^{\infty} A_n$ ,  $A_n \in \mathcal{L}(\mathbb{R})$ , and  $f \in \bar{M}^+(A)$

$$\int_A f = \sum_{n=1}^{\infty} \int_{A_n} f$$

proof: (i) Let  $(\varphi_n)_{n=1}^{\infty}, (\psi_n)_{n=1}^{\infty}$  be increasing sequences of func. from

$S^+(A)$  with  $\lim_{n \rightarrow \infty} \varphi_n = f$  and  $\lim_{n \rightarrow \infty} \psi_n = g$

easy to check

(use lemma), then  $(\varphi_n + \psi_n)_{n=1}^{\infty}$  is an increasing seq. of sample func. □

$$\int_A cf = \lim_{n \rightarrow \infty} \int_A c\varphi_n = \lim_{n \rightarrow \infty} c \int_A \varphi_n \stackrel{(MCT)}{=} c \int_A f$$

$$\int_A (f+g) \stackrel{(MCT)}{=} \lim_{n \rightarrow \infty} \int_A (\varphi_n + \psi_n) = \lim_{n \rightarrow \infty} \int_A \varphi_n + \lim_{n \rightarrow \infty} \int_A \psi_n \stackrel{(MCT)}{=} \int_A f + \int_A g$$

(ii) Let  $g_n = \sum_{k=1}^n f_k$ , then  $g_1 \leq g_2 \leq \dots$

with  $\int_A g_n = \sum_{k=1}^n \int_A f_k$  (by (i) and induction)

and  $\lim_{n \rightarrow \infty} g_n = f$ . By M.C.T.  $\int_A f = \lim_{n \rightarrow \infty} \int_A g_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_A f_k = \sum_{k=1}^{\infty} \int_A f_k$

(iii) if we let  $f_n = f \chi_{A_n}$ , we have

$$\int_A f_n = \int_A f \chi_{A_n} = \int_{A_n} f \quad \text{we use (ii)} \quad \square$$

$S_f^+(A_n)$  extended to a subset  $S_f^+(A)$

$$\tilde{f}(x) = \begin{cases} f & x \in A_n \\ 0 & x \in A \setminus A_n \end{cases}$$

Finally, the Lebesgue integral

Let  $A \in \mathcal{L}(\mathbb{R})$  (usually an interval or all of  $\mathbb{R}$ )

$$\bar{\mathcal{M}}(A) = \{f: A \rightarrow \bar{\mathbb{R}} \mid f \text{ is measurable}\}$$

If  $f \in \bar{\mathcal{M}}(A)$ , define  $f^+ = \max\{f, 0\}$ ,  $f^- = \max\{-f, 0\}$  (pointwise)

$$\text{So } f = \underbrace{f^+ - f^-}_{\text{difference of 2 non-negative functions}}$$

Definition: Let  $f \in \bar{\mathcal{M}}(A)$ , we say that  $f$  is Lebesgue integrable on  $A$  provided both  $\int_A f^+ < +\infty$ ,  $\int_A f^- < +\infty$

In this case, we define the Lebesgue integral of  $f$  on  $A$  by

$$\int_A f = \int_A f^+ - \int_A f^-$$

We let  $\mathcal{L}(A) = \{f: A \rightarrow \mathbb{C} \mid f \text{ is measurable \& Lebesgue integrable}\}$

Lemma (i)  $f \in \mathcal{L}(A)$ , let  $f \in \bar{\mathcal{M}}(A)$

$$\Rightarrow \lambda(\{x \in A : f(x) = \pm\infty\}) = 0$$

$$(ii) \int_A |f| = 0 \Leftrightarrow \lambda(\{x \in A : f(x) \neq 0\}) = 0$$

Proof (i) Suppose  $f \in \mathcal{L}(A)$ , let  $E_+ = \{x \in A : f(x) = +\infty\}$

then for any  $n \in \mathbb{N}$ ,  $n \chi_{E_+} \leq f^+$ , and thus

$$n \lambda(E_+) = \int_A n \chi_{E_+} \leq \int_A f^+ < \infty \text{ by defn. of } f \in \mathcal{L}(A)$$

Thus we must have  $\lambda(E_+) = 0$

similarly,  $E_- = \{x \in A : f(x) = -\infty\}$ ,  $\lambda(E_-) = 0$

and  $E_+ \cup E_- = \{x \in A : f(x) = \pm\infty\}$  is null.

(ii) " $\Rightarrow$ " suppose  $\int_A |f| = 0$  for  $n \in \mathbb{N}$ . Let

$$E_n = \{x \in A : |f(x)| \geq \frac{1}{n}\} = \underbrace{(\bigcup_{k=1}^{\infty} \{x \in A : |f(x)| \geq \frac{1}{n}\})}_{\text{measurable}}$$

We note that  $|f| = f^+ + f^-$  so

$$E_n = (|f|^{-1})([\frac{1}{n}, \infty]) \text{ is measurable, then } \frac{1}{n} \chi_{E_n} \leq |f|$$

$$\text{so } \frac{1}{n} \lambda(E_n) \leq \int_A \frac{1}{n} \chi_{E_n} \leq \int_A |f| = 0 \text{ and then } \lambda(E_n) = 0$$

We have  $\bigcup_{n=1}^{\infty} E_n = \{x \in A : |f(x)| > 0\} = \{x \in A : f(x) \neq 0\}$  and

$$\lambda(E) = 0$$

" $\Leftarrow$ " suppose  $\lambda(\{x \in A : f(x) \neq 0\}) = 0$  let  $\phi \in S_{|f|}^+(A)$

$\varphi$  is simple measurable and  $\varphi \leq |f|$ . Let

$$\varphi(x) = \{\alpha_1 < \dots < \alpha_n\} \quad E_i = \varphi^{-1}(\{\alpha_i\})$$

$$\text{so } \varphi = \sum_{i=1}^n \alpha_i \chi_{E_i} \quad E_i \cap E_j = \emptyset \text{ if } i \neq j$$

If  $\alpha_i > 0$ , then  $\alpha_i \chi_{E_i} \leq \varphi \leq |f|$  so

$$E_i \subset \{x \in A : |f(x)| \geq \alpha_i > 0\} \subset \underbrace{\{x \in A : |f(x)| > 0\}}_{\text{null set}}$$

and hence  $\lambda(E_i) = 0$

We see that  $\alpha_i = 0$  [if  $\lambda(A) > 0$ ] and  $\lambda(E_2) = \dots = \lambda(E_n) = 0$

$$\text{Thus } I_A(\varphi) = \sum_{i=1}^n \alpha_i \lambda(E_i) = 0 \lambda(E_1) + \sum_{i=2}^n \alpha_i \cdot 0 = 0$$

$$\text{and so } \int_A |f| = \sup_{\varphi \in \mathcal{S}_f^+(A)} I_A(\varphi) = 0 \quad \square$$

each = 0

Definition Let  $f, g \in \tilde{M}(A)$ , we say that

" $f = g$  almost everywhere (a.e.) on  $A$ "

$$\text{if } \lambda(\{x \in A : f(x) \neq g(x)\}) = 0$$

Let  $L(A) = \{f : A \rightarrow \mathbb{R} \mid f \text{ is measurable \& Lebesgue integrable}\}$

Corollary (to the lemma above)

If  $f \in \tilde{L}(A)$  then there exists a  $f_0 \in L(A)$  such that

$$f = f_0 \text{ a.e. on } A. \quad \text{Moreover } \int_A |f - f_0| = 0$$

Proof: Let  $f_0(x) = \begin{cases} f(x) & \text{if } f(x) \neq \pm \infty \\ 0 & \text{otherwise} \end{cases}$  By Lemma part (i)

$$f = f_0 \text{ a.e. on } A$$

that  $\int_A |f - f_0| = 0$  is part (ii) of the Lemma  $\square$

Theorem: If  $f, g \in L(A)$ , and  $c \in \mathbb{R}$ , then

$$(i) \quad cf \in L(A) \quad \text{and} \quad \int_A cf = c \int_A f$$

$$(ii) \quad f+g \in L(A) \quad \text{and} \quad \int_A f+g = \int_A f + \int_A g$$

$$(iii) \quad \text{if } |f| \in L(A) \quad \text{and} \quad \left| \int_A f \right| \leq \int_A |f|$$

in fact,  $f \in L(A) \Leftrightarrow f \in M(A)$  and  $|f| \in L(A)$

Proof: (i) straight forward

$$(ii) \quad \text{Note that } (f+g)^+ \leq f^+ + g^+ \quad (f+g)^- \leq f^- + g^-$$

First

which implies that  $\int_A (f+g)^+ \leq \int_A (f^+ + g^+) = \int_A f^+ + \int_A g^+ < \infty$

by defn. of  $L(A)$

Also  $\int_A (f+g)^- < \infty$  and hence  $f+g \in L(A)$

claim: If  $h, k, \phi, \psi \in L^+(A)$  with " $h-k = \phi-\psi$ "

then  $\int_A h - \int_A k = \int_A \phi - \int_A \psi$

Proof:  $h-k = \phi-\psi \Rightarrow h+\psi = \phi+k \Rightarrow \int_A h + \int_A \psi = \int_A (\phi+k)$

$= \int_A (\phi+k) = \int_A \phi + \int_A k$  cor. to M.C.T.

and, since  $\int_A \psi, \int_A k < \infty$  - we subtract from each side.

Now we have

$f^+ - f^- + g^+ - g^- = f+g = (f+g)^+ - (f+g)^-$

$(f^+ + g^+) - (f^- + g^-)$

By claim  $\int_A (f+g) = \int_A (f+g)^+ - \int_A (f+g)^- = \int_A (f^+ + g^+) - \int_A (f^- + g^-)$

Jan 30, 2009 PMath 354

Assign 3. Erratum:

Note on bottom [Don't forget 2.3] on the back.

From last class  $f \in L(A)$

(iii) If  $f \in L(A)$  and  $|\int_A f| \leq \int_A |f|$

Moreover,  $f \in L(A) \Leftrightarrow f \in M(A)$  and  $|f| \in L(A)$

Proof of (iii) First note  $|f| = f^+ + f^-$  Hence  $\max\{f, 0\} = f^+, \max\{-f, 0\} = f^-$

$\int_A |f| = \int_A (f^+ + f^-) = \int_A f^+ + \int_A f^- < \infty$ , so  $|f| \in L(A)$

Now  $|\int_A f| = |\int_A f^+ - \int_A f^-| \leq \int_A f^+ + \int_A f^- = \int_A |f|$

For "Moreover" " $\Rightarrow$ " has just proved  $\int_A f^+ \geq 0, \int_A f^- \geq 0$

" $\Leftarrow$ " If  $|f| \in L(A)$ , we have  $f^+, f^- \leq |f|$

$\Rightarrow \int_A f^+, \int_A f^- \leq \int_A |f| < \infty$  so  $f \in L(A)$  □

Note: If  $E \in \mathcal{P}(\mathbb{R}) \setminus L(\mathbb{R})$ , say  $E \subset (a, b)$  then

$f = \chi_{(a,b) \cap E} \notin \mathcal{M}_E$

We have  $|f| = \chi_{(a,b)} \in \mathcal{M}((a,b))$  but  $f \notin \mathcal{M}((a,b))$

*Mitroy*

Fatou's Lemma: If  $(f_n)_{n=1}^{\infty} \in \bar{\mathcal{M}}^+(A) \subset A \in \mathcal{L}(\mathbb{R})$

then  $\int_A \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int_A f_n$   
pointwise lim inf of functions scalars

Proof: Let  $g_n = \inf_{k \geq n} f_k$ , so  $0 \leq g_1 \leq g_2 \leq \dots$  since  $g_{k+1} = \inf_{n \geq k+1} f_n \geq \inf_{n \geq k} f_n = g_k$   
 and  $\lim_{n \rightarrow \infty} g_n = \liminf_{n \rightarrow \infty} f_n$ , by def'n. of  $\liminf$ .

Hence by M.C.T.

$$\int_A \liminf_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int_A g_n \quad (f)$$

We note that  $g_n \leq f_k$  whenever  $k \geq n$  so

$$\int_A g_n \leq \int_A f_k \text{ for } k \geq n, \text{ Hence}$$

$$\int_A g_n \leq \inf_{k \geq n} \int_A f_k \leq \liminf_{n \rightarrow \infty} \int_A f_k = \liminf_{n \rightarrow \infty} \int_A f_n \quad (ff)$$

The statement follows immediately from (f) and (ff)

Remark " $\leq$ " in Fatou's Lemma may be " $<$ "

Eg:  $A = [0, 1]$

$$f_n = n \chi_{(0, \frac{1}{n})}$$

Check that  $\lim_{n \rightarrow \infty} f_n(x) = 0$  for each  $x \in [0, 1]$

$$\Rightarrow \int_{[0,1]} \liminf_{n \rightarrow \infty} f_n = \int_{[0,1]} \lim_{n \rightarrow \infty} f_n = 0$$

$$\int_{[0,1]} f_n = 1 \Rightarrow \liminf_{n \rightarrow \infty} \int_{[0,1]} f_n = 1$$

Definition Let  $A \in \mathcal{L}(\mathbb{R})$ ,  $(f_n)_{n=1}^{\infty} \in \mathcal{M}(A)$ ,  $f \in \mathcal{M}(A)$ , then we write  $\lim_{n \rightarrow \infty} f_n = f$  a.e. on A if  $f: A \rightarrow \mathbb{R}$  almost everywhere

$N = \{x \in A \mid \lim_{n \rightarrow \infty} f_n(x) \neq f(x), \text{ or doesn't exist}\}$  is a null set.

Notes (i) If  $(f_n)_{n=1}^{\infty} \in \mathcal{M}(A)$  satisfies

$f = \lim_{n \rightarrow \infty} f_n$  a.e. on A, then  $f$  is measurable on A

[ Replace A by  $A \setminus N$ ,  $N$  as in def'n. check that

$f$  is measurable on  $A \setminus N$ . Note that function  $f: N \rightarrow \mathbb{R}$  is automatically measurable. as subset of null sets are null ]

Remark: (i) MCT. and Fatou's Lemma are valid when pointwise convergence is replaced by convergence a.e.

(ii) Advantage of a.e. convergence:

Though  $(f_n)_{n=1}^{\infty} \in M(A)$ , we may have  $f = \lim_{n \rightarrow \infty} f_n \in \bar{M}(A)$  extended real valued

However, if  $f$  is Lebesgue integrable, then we replace  $f$  by  $f_0 \in M(A)$  where  $f = f_0$  a.e. Then  $f_0 = \lim_{n \rightarrow \infty} f_n$  a.e.

Lebesgue's Dominated Convergence Theorem:

If  $(f_n)_{n=1}^{\infty} \subset L(A)$ ,  $f: A \rightarrow \mathbb{R}$ , and  $g \in L^+(A) = \{ \text{Lebesgue integrable } \& g \geq 0 \}$

- st. (i)  $f = \lim_{n \rightarrow \infty} f_n$  a.e on  $A$  and
- (ii)  $|f_n| \leq g$  a.e on  $A$ .

then  $f$  is measurable and Lebesgue integrable with

$$\int_A f = \lim_{n \rightarrow \infty} \int_A f_n$$

Note. often  $g$  is called an integrable majorant for  $(f_n)_{n=1}^{\infty}$

proof: Let  $N = \{x \in A : \lim_{n \rightarrow \infty} f_n(x) \neq f \text{ or } \lim \text{ does not exist}\}$

$$\bigcup_{n=1}^{\infty} \{x \in A : |f_n(x)| \geq g(x)\}$$

So  $N$  is null we replace  $A$  by  $A \setminus N$  and everywhere assumption holds pointwise. relabel as  $A$ .

Now  $f = \lim_{n \rightarrow \infty} f_n$  is measurable, and  $|f| = \lim_{n \rightarrow \infty} |f_n| \leq g$

and hence  $\int_A |f| < \infty$ , so  $f$  is Lebesgue integrable.  $\leq \int_A g < \infty$

Since  $g + f_n \geq 0$  by assumption (ii) and  $g + f = g + \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} (g + f_n)$

We have

$$\begin{aligned}
 (*) \int_A g + \int_A f &= \int_A (g + f) = \int_A \liminf_{n \rightarrow \infty} (g + f_n) \leq \liminf_{n \rightarrow \infty} \int_A (g + f_n) = \liminf_{n \rightarrow \infty} (\int_A g + \int_A f_n) \\
 & \quad \text{by Fatou} \\
 &= \int_A g + \liminf_{n \rightarrow \infty} \int_A f_n
 \end{aligned}$$

Also  $g - f_n \geq 0$  and by (p.s. 2)

$$g - f = g - \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} (g - f_n) = \lim_{n \rightarrow \infty} \inf (g - f_n)$$

Then

$$\begin{aligned} (*) \int_A g - \int_A f &= \int_A (g - f) = \int_A \lim_{n \rightarrow \infty} \inf (g - f_n) \leq \int_A g + \lim_{n \rightarrow \infty} \inf (-\int_A f_n) \\ &= \int_A g - \lim_{n \rightarrow \infty} \sup \int_A f_n \quad \text{Fatou} \end{aligned}$$

we have

$$\begin{aligned} (*) \Rightarrow \lim_{n \rightarrow \infty} \inf \int_A f_n &\geq \int_A f \quad (***) \Rightarrow \int_A g - \lim_{n \rightarrow \infty} \sup \int_A f_n \geq \int_A g - \int_A f \\ &\Rightarrow \lim_{n \rightarrow \infty} \sup \int_A f_n \leq \int_A f \end{aligned}$$

we have

$$\lim_{n \rightarrow \infty} \sup \int_A f_n \leq \int_A f \leq \lim_{n \rightarrow \infty} \inf \int_A f_n \quad \Rightarrow \lim_{n \rightarrow \infty} \int_A f_n \text{ exists and}$$

$$\lim_{n \rightarrow \infty} \int_A f_n = \int_A f \quad \square$$

Feb 2, 2009 PMath 354 Measure Theory

$L_p$ -spaces Let  $A \in \mathcal{L}(\mathbb{R})$  [usually  $A = [a, b]$  or  $A = \mathbb{R}$ ]

Case  $p=1$

Proposition: For  $f \in L(A)$ , define

$$\|f\|_1 = \int_A |f| \quad \text{Then } f, g \in L(A), c \in \mathbb{R}$$

$$(i) \|cf\|_1 = |c| \|f\|_1 \quad (\text{scalar homogeneity})$$

$$(ii) \|f+g\|_1 \leq \|f\|_1 + \|g\|_1 \quad (\text{sub additivity})$$

$$\text{proof (i)} \|cf\|_1 = \int_A |cf| = \int_A |c| |f| = |c| \int_A |f| = |c| \|f\|_1$$

$$(ii) \|f+g\|_1 = \int_A \underbrace{|f+g|}_{\leq |f|+|g|} \leq \int_A (|f|+|g|) = \int_A |f| + \int_A |g| \quad \square$$

How about (iii)  $\|f\|_1 = 0 \stackrel{?}{\Rightarrow} f=0$

we saw earlier that

$$\|f\|_1 = \int_A |f| = 0 \Leftrightarrow f=0 \text{ ae. on } A$$

we define an equivalence relation on  $L(A)$ :

$$f \sim g \Leftrightarrow f=g \text{ ae. on } A$$

and we define

$$L_1(A) = L(A) / \sim$$

Hence we think of  $L_1(A)$  as the space of integrable functions on  $A$  where we agree  $f=g$  if  $f=g$  ae. on  $A$

Then  $\|\cdot\|_1$  defines a norm on  $L_1(A)$

warning since  $\{x\}$  is null for  $x \in A$ , the value " $f(x)$ " for  $f \in L_1(A)$  is meaningless.

consequence:

convergence in  $L_1$ : if  $(f_n)_{n=1}^\infty \in L_1(A)$  and  $f \in L_1(A)$  st.  $\lim_{n \rightarrow \infty} f_n = f$  a.e. and there is  $g \in L_1^+(A)$  st.

if  $|f_n| \leq g$ , then we can conclude

$$\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$$

First note  $|f| = \lim_{n \rightarrow \infty} |f_n| \leq g$  a.e. so

proof:  $\|f_n - f\|_1 \leq \|f_n\|_1 + \|f\|_1 \leq 2g$  so we can use LDCT to

show that

$$\|f_n - f\|_1 = \int_A |f_n - f| \xrightarrow{n \rightarrow \infty} \int_A 0 = 0 \text{ since } f_n \xrightarrow{a.e.} f$$

Question: If  $\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$ , does this imply  $\lim_{n \rightarrow \infty} f_n = f$  a.e.

Answer: No.

Example:  $A = [0, 1]$

$$\text{Let } f_1 = \chi_{[0, \frac{1}{2}]} \quad f_2 = \chi_{[\frac{1}{2}, 1]}$$

$$f_3 = \chi_{[0, \frac{1}{3}]} \quad f_4 = \chi_{[\frac{1}{3}, \frac{2}{3}]} \quad f_5 = \chi_{[\frac{2}{3}, 1]}$$

$$f_6 = \chi_{[0, \frac{1}{4}]} \quad \dots \quad f_9 = \chi_{[\frac{3}{4}, 1]} \quad \dots$$

check that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} |f_n - 0| = \lim_{n \rightarrow \infty} \int_{[0,1]} f_n = 0$$

$\lim_{n \rightarrow \infty} f_n(x)$  exists for no  $x \in [0, 1]$

In particular,  $\lim_{n \rightarrow \infty} |f_n - 0|$  D.N.E a.e on  $[0, 1]$   
Does NOT Exist

Case  $1 < p < \infty$

Definition: We define the conjugate index to  $p$  as the number  $q$  which satisfies  $\frac{1}{p} + \frac{1}{q} = 1$

[if  $p=1$ , we set  $q=\infty$ ; if  $p=\infty$ , we let  $q=1$ ]

Now we define for  $f \in M(A)$

$$\|f\|_p = \left( \int_A |f|^p \right)^{1/p}$$

could be  $\infty$ .

We define  $L_p(A) = \{ f \in M(A) : |f|^p \in L(A) \} / \sim$  a.e.

Lemma: If  $1 < p < \infty$ ,  $q$  is the conjugate index,  $\int_A |f|^p < \infty$  and  $a, b \in [0, \infty)$  then

$$a \cdot b < \frac{a^p}{p} + \frac{b^q}{q}$$

Moreover "=" holds only when  $a^p = b^q$

proof: If  $ab = 0$ , we are done; we assume  $ab > 0$  hereafter.

Let  $0 < \alpha < 1$  Let

$$\varphi: (0, \infty) \rightarrow \mathbb{R} \quad \varphi(t) = \alpha t - t^\alpha$$

Then we have that

$$\varphi'(t) = \alpha - \alpha t^{\alpha-1} = \alpha \left[ 1 - t^{\alpha-1} \right] = \alpha \left[ 1 - \frac{1}{t^{1-\alpha}} \right]$$

and we have  $\varphi'(t) < 0$  if  $0 < t < 1$

and  $\varphi'(t) > 0$  if  $t > 1$

and  $\varphi'(t) = 0 \Leftrightarrow t = 1$

Thus by application of MVT (Mean Value Theorem) we see that

$$\varphi(t) = \alpha t - t^\alpha \geq \varphi(1) = \alpha - 1 \quad \text{with "=" only when } t=1$$

$$\text{Now set } t = \frac{a^p}{b^q} \Rightarrow t^\alpha \neq \alpha t + (1-\alpha)$$

$$\left( \frac{a^p}{b^q} \right)^\alpha \leq \alpha \frac{a^p}{b^q} + (1-\alpha) \Rightarrow$$

$$a^{\alpha p} b^{q(1-\alpha)} = \alpha a^p + (1-\alpha) b^q$$

Now let  $\alpha = \frac{1}{p}$  so  $1-\alpha = \frac{1}{q}$  Note "=" holds only when  $a^p = b^q$   $\square$

Hölder's inequality If  $f \in L_p(A)$  [ $1 < p < \infty$ ] and  $g \in L_q(A)$  [ $q$  is conjugate index to  $p$ ].

then  $f \cdot g$  (pointwise product a.e.) is in  $L_1(A)$  and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Moreover, "=" holds  $\Leftrightarrow \|g\|_q |f| = \|f\|_p |g|$  a.e.

proof: If  $\|f\|_p \|g\|_q = 0$  then

$fg = 0$  a.e. so the inequality holds. Assume, hereafter,  $\|f\|_p \|g\|_q > 0$

$$\text{Let for a.e. } x \in A, \quad a(x) = \frac{|f(x)|}{\|f\|_p} \quad b(x) = \frac{|g(x)|}{\|g\|_q}$$

Then by the Lemma above, for a.e.  $x \in A$

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} = a(x)b(x) \leq \frac{a(x)^p}{p} + \frac{b(x)^q}{q} = \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}$$

Note that since  $f, g$  are each measurable (a.e.) we have  $fg$  is measurable, hence so too is  $|fg|$

Thus

$$\frac{1}{\|f\|_p \|g\|_q} \int_A |fg| \leq \int_A \left( \frac{|f|^p}{p\|f\|_p^p} + \frac{|g|^q}{q\|g\|_q^q} \right) = \frac{\|f\|_p^p \|g\|_q}{p\|f\|_p^p} + \frac{\|g\|_q^q \|f\|_p}{q\|g\|_q^q} = 1$$

$$\Rightarrow \frac{\int_A |fg|}{\|f\|_p \|g\|_q} \leq \frac{\|f\|_p^p}{p\|f\|_p^p}$$

$$\|fg\|_1 = \int_A |fg| \leq \|f\|_p \|g\|_q$$

To see when "=" holds, use the "=" case from the Lemma and follow through □

Feb 4, 2009 PMath 354

Holder's Inequality If  $1 < p < \infty$ ,  $q$ : conjugate index ( $\frac{1}{p} + \frac{1}{q} = 1$ )

$A \subseteq \mathcal{L}(\mathbb{R})$ ,  $f \in L_p(A)$ ,  $g \in L_q(A)$  then  $fg \in L_1(A)$

with  $\|fg\|_1 = \int_A |fg| \leq \left( \int_A |f|^p \right)^{1/p} \left( \int_A |g|^q \right)^{1/q} = \|f\|_p \|g\|_q$

Moreover, "=" holds only if  $\|g\|_q^q \|f\|_p = \|f\|_p^p \|g\|_q^q$  a.e.

Minkowski's Inequality.

If  $1 < p < \infty$ ,  $f, g \in L_p(A)$  [ $A \in \mathcal{L}(\mathbb{R})$ ]

then  $f+g \in L_p(A)$  and  $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

Moreover "=" holds only if there are  $C_1, C_2 > 0$  s.t.  $C_1 f = C_2 g$  a.e.

Corollary:  $\|\cdot\|_p$  is a norm on  $L_p(A)$

Proof: 1-Scalar homogeneity  $\|cf\|_p = |c| \|f\|_p$  is obvious

non-negativity  $\|f\|_p = 0 \Leftrightarrow \int_A |f|^p = 0 \Leftrightarrow |f|^p = 0$  a.e.  $\Leftrightarrow f = 0$  a.e.

subadditivity  $\|f+g\|_p \leq \|f\|_p + \|g\|_p$  Minkowski's inequality □

Proof of Minkowski's Ineq. First,  $f, g \in L_p(A)$  we have

$$|f+g|^p \leq (2 \max\{|f|, |g|\})^p = 2^p \max\{|f|^p, |g|^p\} \leq 2^p (|f|^p + |g|^p)$$

so  $\int_A |f+g|^p \leq \int_A 2^p (|f|^p + |g|^p) = 2^p \left( \int_A |f|^p + \int_A |g|^p \right) < \infty$   
by assumptions



check that this condition implies equality holds.

$$\|f+g\|_p = \|f\|_p \|g\|_p \quad (11)$$

Complete normed spaces:

(i) If  $X$  is a  $\mathbb{R}$ -vector space with norm  $\|\cdot\|$ . Then  $(X, \|\cdot\|)$  is complete if every Cauchy sequence in this norm converges to a point in  $X$ .

I.e. if  $(x_n)_{n=1}^\infty$  st. for  $\epsilon > 0$  there is  $n_\epsilon \in \mathbb{N}$ , so,  $n, m \geq n_\epsilon \Rightarrow \|x_n - x_m\| < \epsilon$  (Cauchy). implies there exists  $x \in X$  st.  $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$  " $\lim_{n \rightarrow \infty} x_n = x$ "

(ii) If  $(X, \|\cdot\|)$  is a normed vector space, then it is complete if and only if for each sequence  $(x_n)_{n=1}^\infty \subset X$ ,  $\sum_{n=1}^\infty \|x_n\| < \infty \Rightarrow \sum_{n=1}^\infty x_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n$  exists in  $X$ .

(iii) If  $(x_n)_{n=1}^\infty$  is a Cauchy sequence in  $X$ , then there is a subsequence  $(x_{n_k})_{k=1}^\infty$  [ $n_1 < n_2 < n_3 < \dots$ ] st.

$$\|x_{n_k} - x_{n_{k+1}}\| < \frac{1}{2^k}$$

Theorem  $A \in \mathcal{L}(\mathbb{R})$  say  $A = [a, b]$  or  $\mathbb{R}$ . If  $1 \leq p < \infty$ , then  $(L_p(A), \|\cdot\|_p)$  is complete.

proof: Let  $(f_n)_{n=1}^\infty \subset L_p(A)$  be a Cauchy sequence for  $\|\cdot\|_p$ . By dropping to a subsequence, and relabelling, we may assume

$$\|f_{n+1} - f_n\|_p < \frac{1}{2^n}$$

[ Thus if we find a limit for the "sequence", we obtain one for the whole sequence ]

We let  $f = f_1 + \sum_{n=1}^\infty (f_{n+1} - f_n)$  a.e.

we must show that the limit is  $\mathbb{R}$ -valued a.e. which will be true if  $|f| \in L_1(A)$

For each  $k \in \mathbb{N}$ , let  $g_k = |f_1| + \sum_{n=1}^k |f_{n+1} - f_n|$

Then  $g_k \geq 0$  a.e. and  $g_1 \leq g_2 \leq \dots$  with a.e. limit

$$g = \lim_{k \rightarrow \infty} g_k = |f_1| + \sum_{n=1}^\infty |f_{n+1} - f_n|$$

By Minkowski's Ineq. we have

$$\|g_k\|_p \leq \|f_1\|_p + \sum_{n=1}^k \|f_{n+1} - f_n\|_p \leq \|f_1\|_p + \sum_{n=1}^k \frac{1}{2^n}$$

$$\leq \|f_1\|_p + 1$$

and Thus  $\int_A g_k^p = \|g_k\|_p^p \leq (\|f_1\|_p + 1)^p$  also  $g_1^p \leq g_2^p \leq \dots$

by M.C.T.  $g^p \in L_1(A)$  with  $\int_A g^p = \lim_{k \rightarrow \infty} \int_A g_k^p$

Hence  $g$  is  $\mathbb{R}$ -valued a.e. now

$$|f|^p = \left| f_1 + \sum_{n=1}^{\infty} (f_{n+1} - f_n) \right|^p \leq \left| \|f_1\|_p + \sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_p \right|^p$$

$$\leq g^p$$

So  $|f|^p \in L_1(A)$  and  $f$  is  $\mathbb{R}$ -valued a.e. and  $f \in L_p(A)$

Now for  $n \in \mathbb{N}$

$$\|f_n - f\|_p = \left\| f_n - \left( f_1 + \sum_{k=1}^{\infty} (f_{k+1} - f_k) \right) \right\|_p$$

$$= \left\| f_n - (f_1 + (f_2 - f_1) + (f_3 - f_2) + \dots + (f_{n+1} - f_n) + \dots) \right\|_p$$

$$= \left\| \sum_{k=n+1}^{\infty} (f_{k+1} - f_k) \right\|_p \leq \sum_{k=n+1}^{\infty} \|f_{k+1} - f_k\|_p$$

$$\leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} \xrightarrow{n \rightarrow \infty} 0 < \frac{1}{2^n}$$

So  $\lim_{n \rightarrow \infty} f_n = f$

Feb 6, 2009 PMath 354 Measure Theory

(Case  $p = \infty$ )  $A \in \mathcal{L}(\mathbb{R})$  [usually  $A = \mathbb{R}, [a, b]$ ]

A function  $f \in \mathcal{M}(A)$  is essentially bounded on  $A$  if there is  $C \geq 0$  s.t.

$$\lambda \left\{ x \in A : |f(x)| > C \right\} = 0$$

We define for such  $f$   $\|f\|_{\infty} = \inf \{ C \geq 0 : C \text{ is essential bound for } f \}$

$$\|f\|_{\infty} = \text{ess sup}_{x \in A} |f(x)|$$

Note  $\|f\|_{\infty}$  is an essential bound. Indeed

$$E_n = \{ x \in A : |f(x)| \geq \|f\|_{\infty} + \frac{1}{n} \}$$

then  $\lambda(E_n) = 0$  since  $\|f\|_{\infty} + \frac{1}{n}$  must be an essential bound

Thus  $E = \bigcup_{n=1}^{\infty} E_n = \{x \in A : |f(x)| > \|f\|_{\infty}\}$  is a null set

We define  $L_{\infty}(A) = \{f \in M(A) : f \text{ is essentially bounded}\} / \sim_{ae}$

where  $f \sim_{ae} g \iff f = g \text{ ae on } A$

Proposition: If  $f, g \in L_{\infty}(A), c \in \mathbb{R}$ , then

(i)  $\|f\|_{\infty} = 0 \iff f = 0 \text{ ae}$

(ii)  $\|cf\|_{\infty} = |c| \|f\|_{\infty}$

(iii)  $\|f+g\|_{\infty} \leq \|f\|_{\infty} + \|g\|_{\infty}$

Thus  $L_{\infty}$  is a vector space and  $\|\cdot\|_{\infty}$  is a norm.

proof: (i)(ii) straight-forward exercise.

(iii) Let us first see that, simultaneously,

$\|f\|_{\infty} + \|g\|_{\infty}$  is an essential bound for  $|f+g|$

and hence  $\|f\|_{\infty} + \|g\|_{\infty} \geq \|f+g\|_{\infty}$

"inf" of essential bounds

$\{x \in A : |f(x) + g(x)| > \|f\|_{\infty} + \|g\|_{\infty}\}$

$\{x \in A : |f(x)| + |g(x)| > \|f\|_{\infty} + \|g\|_{\infty}\}$

$|f| + |g| \geq |f+g|$

$\{x \in A : |f(x)| > \|f\|_{\infty}\} \cup \{x \in A : |g(x)| > \|g\|_{\infty}\}$

null

null

and hence  $\{x \in A : |f(x) + g(x)| > \|f\|_{\infty} + \|g\|_{\infty}\}$  is null.  $\square$

Note: If  $\lambda(A) < \infty$  (say  $A = [a, b]$ ), then for

$f \in L_{\infty}(A)$  we have  $\|f\|_{\infty} = \lim_{p \rightarrow \infty} \|f\|_p$  (Through calculus)

Theorem:  $(L_{\infty}(A), \|\cdot\|_{\infty})$  is complete, hence it is a Banach space.

Proof: Let  $(f_n)_{n=1}^{\infty} \subset L_{\infty}(A)$  be a Cauchy sequence.

By dropping to subsequence and relabelling if necessary, we may suppose

$\|f_{n+1} - f_n\|_{\infty} < \frac{1}{2^n}$

Then we let

$$f = f_1 + \sum_{n=1}^{\infty} (f_{n+1} - f_n)$$

we ~~find~~ establish  $f \in L_{\infty}(A)$

$$\text{Let } F = \{x \in A \mid |f_n(x)| > \|f_1\|_{\infty}\} \text{ and } E_n = \{x \in A, |f_{n+1}(x) - f_n(x)| > \|f_{n+1} - f_n\|_{\infty}\}$$

Then  $F, E_n$  are null for  $n \in \mathbb{N}$  so

$$E = F \cup \bigcup_{n=1}^{\infty} E_n \text{ is null too.}$$

Thus for  $x \in A \setminus E$  we have

$$\begin{aligned} |f(x)| &\leq |f_1(x) + \sum_{n=1}^{\infty} (f_{n+1}(x) - f_n(x))| \leq |f_1(x)| + \sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)| \\ &\leq \|f_1\|_{\infty} + \sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_{\infty} \leq \|f_1\|_{\infty} + 1 \\ &\qquad\qquad\qquad < \frac{1}{2^n} \end{aligned}$$

thus  $\|f_1\|_{\infty} + 1$  is an essential bound for  $f$ , so  $f$  is essentially bounded. Clearly, on  $A \setminus E$ ,  $f \in M(A \setminus E)$

So indeed,  $f \in L_{\infty}(A)$

That  $\lim_{n \rightarrow \infty} \|f - f_n\|_{\infty} = 0$  is proved exactly as in the  $L_p(A)$ -case,  $1 \leq p < \infty$   $\square$

Containment relations.

Let  $A = [a, b]$  a bounded interval,  $\lambda([a, b]) = b - a$

Proposition: Let  $1 \leq p < r < \infty$ . Then  $L_r([a, b]) \subset L_p([a, b])$

with  $\|f\|_p \leq \|f\|_r (b-a)^{\frac{r-p}{rp}}$  for  $f \in L_r([a, b])$

Proof: Let  $f \in L_r([a, b])$ , then  $|f|^p \in L_{\frac{r}{p}}([a, b])$

Then by Holder's Ineq.

$$\int_{[a, b]} |f|^p = \int_{[a, b]} |f|^p \cdot 1 \leq \| |f|^p \|_{\frac{r}{p}} \|1\|_q \text{ where } q \text{ is conjugate to } \frac{r}{p}$$

$$\text{i.e. } \frac{1}{\frac{r}{p}} + \frac{1}{q} = 1 \Leftrightarrow \frac{p}{r} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = 1 - \frac{p}{r} = \frac{r-p}{r}$$

$$\|f\|_p \leq \| |f|^p \|_p^{1/p} = \left( \int_a^b (|f|^p)^{1/p} \right)^{p/p} = \left( \int_a^b |f| \right)^{1/p} = \|f\|_1^{1/p}$$

$$= (\|f\|_1)^{1/p} (b-a)^{1 - 1/p}$$

$$= \|f\|_1^{1/p} (b-a)^{1/p}$$

Remarks: (i) Note  $L_\infty([a,b]) \subset L_p([a,b])$  for any  $1 \leq p < \infty$   
 (Assignment #4)

(ii) Let  $A = \mathbb{R}, [0, \infty)$  Do we have  $1 \leq p < r < \infty$

$$L_r(A) \subset L_p(A), L_p(A) \subset L_r(A)?$$

Note,  $L_r(A) = L_r([0, \infty)) \not\subset L_p([0, \infty))$

Example.  $f(x) = \max\{1, \frac{1}{x^p}\}$   $x \in [0, \infty)$

Then  $f \in L_r([0, \infty))$  but  $f \notin L_p([0, \infty))$

Note  $L_p([0,1]) \not\subset L_r([0,1]) \Rightarrow L_p([0, \infty)) \not\subset L_r([0, \infty))$   
 exercise.

Let  $f(x) = \frac{1}{x^r}$  a.e.

Then  $f \in L_p([0,1])$ ,  $f \notin L_r([0,1])$

Feb 9, 2009 PMath 354 Assignment 3 may hand in on Friday  
 Midterm Feb 26, 7:00 ~ 8:30 pm Rm. TBA  
 Thursday

Comparisons between Riemann & Lebesgue integral

On Assignment 3: It is shown

$$f: [a,b] \rightarrow \mathbb{R} \text{ is Riemann integrable} \Rightarrow f \text{ is Lebesgue integrable}$$

$$\& \int_{[a,b]} f = \int_a^b f$$

(Lebesgue) (Riemann)

Also

$f: [a,b]$  is Riemann integrable  $\Rightarrow \lambda(\{x \in [a,b]; f \text{ is not continuous at } x\}) = 0$   
 &  $f$  is bounded.

The converse is true. [The proof is tricky]

fact to be proved: If the set of points of discontinuity is null.

We need  $E_n = \{x \in [a,b]; \text{ for each } \delta > 0, \text{ there are } \gamma, z \in (x-\delta, x+\delta) \cap [a,b] \text{ s.t. } |f(\gamma) - f(z)| > \frac{1}{n}\}$   
 satisfies that  $\inf \sum_{k=1}^n \lambda(I_k) = 0$   $\{I_k\} \subset \mathbb{R}$  open interval,  $k \in \mathbb{N}$

## Improper Riemann Integral

If  $f: [0, \infty) \rightarrow \mathbb{R}$ , then we say  $f$  is "improper Riemann" integrable if

• for any  $b > 0$ ,  $f$  is Riemann integrable  $[0, b]$

•  $\lim_{b \rightarrow \infty} \int_0^b f$  exists.

Proposition: If  $f: [0, \infty) \rightarrow \mathbb{R}$  is "improper - Riemann" integrable as above, and

$f(x) \geq 0$  for  $x \in [0, \infty)$ , then

$f$  is Lebesgue integrable on  $[0, \infty)$  with

$$\int_{[0, \infty)} f = \int_0^{\infty} f = \lim_{b \rightarrow \infty} \int_0^b f$$

Proof: From Assignment 3,  $f$  is Lebesgue integrable on each interval  $[0, b]$  with

$$\int_{[0, b]} f = \int_0^b f$$

Define  $f_n = f \chi_{[0, n]}$  Then  $f_1 \leq f_2 \leq \dots$  and

$$\lim_{n \rightarrow \infty} f_n = f \text{ (pointwise)}$$

Hence by MCT, we have

$$\int_{[0, \infty)} f = \lim_{n \rightarrow \infty} \int_{[0, \infty)} f_n = \lim_{n \rightarrow \infty} \int_{[0, n]} f \chi_{[0, n]} = \lim_{n \rightarrow \infty} \int_{[0, n]} f = \lim_{n \rightarrow \infty} \int_0^n f = \int_0^{\infty} f \quad \square$$

Note: (i) If we do not assume  $f \geq 0$ , this may fail. The improper Riemann integral may exist, but  $f$  is not Lebesgue integrable on  $[0, \infty)$

Exercise: [Hint:  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$  converges, but  $\sum_{n=0}^{\infty} \frac{1}{n}$  does not]

(ii) There is an improper Riemann integral for unbounded  $f: (0, 1] \rightarrow \mathbb{R}$

where  $f$  is Riemann integrable on each  $[a, 1]$   $a > 0$

The same facts regarding Lebesgue integrability hold.

### Some functional analysis on $L^p$ -spaces.

Definitions: If  $X, Y$  are Banach spaces, a linear operator  $T: X \rightarrow Y$  is bounded provided that

$$\|T\| = \sup \{ \|Tx\|_Y : \|x\|_X \leq 1 \} < \infty$$

If  $Y = \mathbb{R}$ , linear operators  $f: X \rightarrow \mathbb{R}$  are called linear functionals.

In this context, we use  $\|f\|_* = \|f\|$ .  
we denote the space of all bounded linear functionals on  $X$  by  $X^*$   
and call it the dual. --- ①

Note:  $\|T\| = \sup \{ \|Tx\|_y : \|x\|_x \leq 1 \} = \sup \{ \|Tx\|_y : \|x\|_x < 1 \}$  --- ②

clearly, ②  $\leq$  ①, To see that ①  $\leq$  ②

Note for  $\|x\|_x \leq 1$ , then for  $0 < \eta < 1$  we have

$\|\eta x\|_x = \eta \|x\|_x < 1$  and we have

$\|x\| = \sup_{0 < \eta < 1} \|\eta x\|$   
 $\|x\| = \sup_{\|x\| < 1} \|x\|$

Thus ①  $\sup \{ \|Tx\|_y : \|x\|_x \leq 1 \} =$   
 $\sup \{ \|Tx\|_y : \|x\|_x \leq \eta, 0 < \eta < 1 \} \leq$   
 $\eta \sup \{ \|Tx\|_y : \|x\|_x \leq 1 \} = \eta \|T\|$

proposition: Let  $X, Y$  be Banach spaces,

$T: X \rightarrow Y$  be a linear operator. then TFAE

- (i)  $T$  is continuous.
- (ii)  $T$  is bounded i.e.  $\|T\| < \infty$
- (iii)  $T$  is Lipschitz, moreover, Lipschitz constant  $\|T\|$ .

proof: (i)  $\Rightarrow$  (ii) If  $T$  is continuous on  $X$ , then  $T$  is continuous at  $0_x$   
( $\in X$ ) since

$B_1(0_y) = \{ y \in Y : \|y - 0_y\| < 1 \}$  is open in  $Y$  and  $T(0_x) = 0_y$

By the continuity, there is a neighbourhood

$B_\delta(0_x) = \{ x \in X : \|x - 0_x\|_x < \delta \}$  st.

$T(B_\delta(0_x)) \subset B_1(0_y)$ , zf  $\|x\|_x < 1$

then  $\|\delta x\|_x = \delta \|x\|_x < \delta$  so  $\delta x \in B_\delta(0_x)$

thus  $\delta \|Tx\|_y = \|T(\delta x)\|_y < 1$  and thus

$\|T\| = \sup \{ \|Tx\|_y : \|x\|_x < \frac{1}{\delta} \} \leq \frac{1}{\delta} < \infty$

so  $T$  is bounded.

(ii)  $\Rightarrow$  (iii) First note that  $\|Tx\|_y \leq \|T\| \|x\|_x$  for  $x \in X$

Indeed, this holds  $\forall 0_x$

if  $x \neq 0_x$ , we have  $\left\| \frac{1}{\|x\|_x} x \right\|_x = 1 \leq 1$

and thus  $\frac{1}{\|x\|_x} \|Tx\|_y = \left\| T \left( \frac{x}{\|x\|_x} \right) \right\|_y \leq \|T\|$

So we are done.

Now for  $x, x' \in X$  we have

$$\|Tx - Tx'\|_y = \|T(x - x')\|_y \leq \|T\| \|x - x'\|_x$$

Which shows the Lipschitz property, with a Lipschitz bound  $\|T\|$ .

[Note, it is routine to check that  $\|T\|$  is the smallest  $C \geq 0$ , st.

$$\|Tx - Tx'\|_y \leq C \|x - x'\|_x ]$$

(iii)  $\Rightarrow$  (i) Lipschitz property  $\Rightarrow$  uniformly continuous  $\Rightarrow$  continuity  $\square$

Feb 11, 2009 PMath 354 Measure Theory

$X$  Banach space. Linear functional is  $\underbrace{[ X \rightarrow \mathbb{R} ]}_{\text{a map}}$

st.  $[x+y] = [x] + [y]$   $[ \alpha x ] = \alpha [x]$   $x, y \in X, \alpha \in \mathbb{R}$

We say  $[$  is bounded if  $\|[ ]\|_* = \sup \{ |[x]| : x \in X, \|x\| \leq 1 \} < \infty$

Theorem: Let  $1 < p < \infty$   $A \in \mathcal{L}(\mathbb{R})$   $q$  be the conjugate index and

$g \in L_q(A)$  Then

Eg:  $L_p(A) \rightarrow \mathbb{R}$ , given by  $[g](f) = \int_A fg$  defines a bounded linear functional

on  $L_p(A)$  with  $\|[g]\|_* = \|g\|_q$

Fact  $\leadsto$  fact, every bounded linear functional

$[ L_p(A) \rightarrow \mathbb{R} ]$  is of the form  $[ = [g]$  for some choice of  $g \in L_q(A)$

[proof requires Radon-Nikodym Thm. PMath 454]

$\leadsto$  functional analysis, we say  $L_q(A)$  is the dual space of  $L_p(A)$  written

$$L_p(A)^* \cong L_q(A)$$

proof: First, if  $g \in L^q(A)$   $f \in L^p(A)$  then by Hölder's inequality

we see that  $fg \in L^1(A)$ , and hence

$fg$  is integrable so  $[g(f)] = \int_A fg$  makes sense.

It is obvious that  $[g]$  is linear. Also again by Hölder's inequality we have for  $f \in L^p(A)$

$$|[g(f)]| = |\int_A fg| \stackrel{\text{Hölder}}{\leq} \int_A |fg| \leq \|f\|_p \|g\|_q$$

Hence we have

$$\|[g]\|_* = \sup \{ |[g(f)]| : f \in L^p(A), \|f\|_p \leq 1 \}$$

$$\leq \sup \{ \|f\|_p \|g\|_q : f \in L^p(A), \|f\|_p \leq 1 \} \leq 1 \cdot \|g\|_q = \|g\|_q$$

Thus  $[g]$  is bounded with  $\|[g]\|_* \leq \|g\|_q$

Let us see that  $\|[g]\|_* \geq \|g\|_q$ .

Let us take a cue from the "=" case of Hölder inequality

Let us choose  $f = C |g|^{q/p} \text{sign } g$  where  $C \geq 0$

$$[\text{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \quad \text{sign} : \mathbb{R} \rightarrow \mathbb{R} \quad \text{Borel-meas.}]$$

$\Rightarrow \text{sign } g$  is meas. if  $g$  is meas.

First let us check that  $f \in L^p(A)$

$$\begin{aligned} \|f\|_p^p &= \int_A |f|^p = \int_A |C |g|^{q/p} \text{sign } g|^p = C^p \int_A |g|^q |\text{sign } g|^p \\ &= C^p \int_A |g|^q = C^p \|g\|_q^q < \infty \end{aligned}$$

Hence  $f \in L^p(A)$  with  $\|f\|_p = C \|g\|_q^{q/p}$

If we choose  $C = \frac{1}{\|g\|_q^{q/p}}$  [requires  $g \neq 0$  ae]

then we obtain  $\|f\|_p = 1$  [if  $g = 0$  ae. then  $[g(f)] = 0$ .

$$\text{and hence } \|[g]\|_* = 0 \\ = \|g\|_q$$

Thus we have

$$\|Tg\|_* = \sup \{ |Tg(f)| : f \in L_p(A) \text{ } \|f\|_p \leq 1 \}$$

$$\geq |Tg(\frac{1}{\|g\|_q^{\frac{1}{p}}}|g|^{\frac{q}{p}} \text{sgn } g)|$$

$$= |\int_A \frac{1}{\|g\|_q^{\frac{1}{p}}} |g|^{\frac{q}{p}} (\text{sgn } g) g| = \frac{1}{\|g\|_q^{\frac{1}{p}}} \underbrace{|\int_A |g|^{\frac{q}{p}+1}}_e$$

$$\frac{q}{p} + 1 = \frac{q+p}{p} = q(\frac{1}{p} + \frac{1}{q}) = q$$

$$q - \frac{q}{p} = q(1 - \frac{1}{p}) = q(\frac{1}{q}) = 1$$

$$= \|e\|_q^{q - \frac{q}{p}} = \|g\|_q$$

Hence  $\|Tg\|_* \geq \|g\|_q$  □

Theorem: Let  $g \in L_\infty(A)$   $A \in L(\mathbb{R})$

Define  $Tg: L_1(A) \rightarrow \mathbb{R}$  by  $Tg(f) = \int_A fg$

Then  $Tg$  is a bounded linear functional with

$$\|Tg\|_* = \|g\|_\infty$$

Proof: If  $f \in L_1(A)$   $g \in L_\infty(A)$ , we have

$$\int_A |fg| \leq \int_A |f| \|g\|_\infty = \|g\|_\infty \int_A |f| = \|g\|_\infty \|f\|_1,$$

$$|g(x)| \leq \|g\|_\infty \text{ for a.e. } x \text{ in } A$$

$$\Rightarrow |f(x)g(x)| = |f(x)||g(x)| \leq |f(x)| \|g\|_\infty \quad \left( \begin{array}{l} \text{(Sometimes also called} \\ \text{Hölder's 2nd)} \end{array} \right)$$

Thus  $fg \in L_1(A)$  and we have

$$|\int_A fg| \leq \int_A |fg| \leq \|f\|_1 \|g\|_\infty$$

As in the proof, above, it follows that  $\|Tg\|_* \leq \|g\|_\infty$

Let us see that  $\|Tg\|_* \geq \|g\|_\infty$  First of all, if  $g=0$  a.e.

then this is trivial. Thus we assume  $\|g\|_\infty > 0$

Let  $B = \{x \in A : |g(x)| > 0\}$

so  $\lambda(B) > 0$

statement of Thm  
 $0 < \lambda(A) < \infty$

It is possible that  $\lambda(B) = \infty$  we find as in Assign. 2, measurable <sup>49</sup>  
 we define  $B_1 \subset B_2 \subset \dots \subset A$  st.  $B = \bigcup_{n=1}^{\infty} B_n$  and  $\lambda(B_n) < \infty$

let  $f_n = \frac{1}{\lambda(B_n)} \text{sgn } g \chi_{B_n}$  we let  $B_n = \{x \in A : |g(x)| > \|g\|_{\infty} - \frac{1}{n}\}$

Thus  $\|f_n\|_1 = \int_A \frac{1}{\lambda(B_n)} |\text{sgn } g \chi_{B_n}| = \frac{1}{\lambda(B_n)} \int_A \chi_{B_n} = 1$  so  $\lambda(B_n) \neq 0$

$$\|Tg\|_{\infty} = \sup \{ |Tg(f)| : f \in L_1(A), \|f\|_1 \leq 1 \}$$

$$\geq |Tg(f_n)| = \left| \int_A \frac{1}{\lambda(B_n)} \text{sgn } g \chi_{B_n} g \right|$$

$$= \frac{1}{\lambda(B_n)} \int_{B_n} |g| \geq \frac{1}{\lambda(B_n)} \int_{B_n} (\|g\|_{\infty} - \frac{1}{n}) = \|g\|_{\infty} - \frac{1}{n}$$

$|g(x)| > \|g\|_{\infty} - \frac{1}{n} \quad x \in B_n$

Since  $n \in \mathbb{N}$  is arbitrary,  $\|Tg\|_{\infty} \geq \|g\|_{\infty}$  □

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Theorem: Let  $1 \leq p < \infty$ ,  $A \in \mathcal{L}(\mathbb{R})$ ,  $\phi \in L_{\infty}(A)$

Define  $M_{\phi}: L_p(A) \rightarrow L_p(A)$  by  $M_{\phi}f = \phi f$

Then  $M_{\phi}$  is a linear operator with

$$\|M_{\phi}\| = \|\phi\|_{\infty}$$

Proof: We must first check that  $\phi f \in L_p(A)$  for  $f \in L_p(A)$

$$|\phi f|^p \leq |\phi|^p |f|^p \leq \|\phi\|_{\infty}^p |f|^p$$

$\int_{ae} \leq \|\phi\|_{\infty}^p$

$$\Rightarrow \int_A |\phi f|^p \leq \int_A \|\phi\|_{\infty}^p |f|^p = \|\phi\|_{\infty}^p \int_A |f|^p < \infty \Rightarrow \phi f \in L_p(A)$$

It is straightforward that  $M_{\phi}$  is linear.

$$\text{Also, } \|M_{\phi}f\|_p = \left( \int_A |\phi f|^p \right)^{1/p} \leq \left( \|\phi\|_{\infty}^p \int_A |f|^p \right)^{1/p} = \|\phi\|_{\infty} \|f\|_p$$

from above

$$\text{and hence } \|M_{\phi}\| = \sup \{ \|M_{\phi}f\|_p : f \in L_p(A), \|f\|_p \leq 1 \} = \|\phi\|_{\infty}$$

$$\leq \sup \{ \|\phi\|_{\infty} \|f\|_p : f \in L_p(A), \|f\|_p \leq 1 \} = \|\phi\|_{\infty}$$

□

Let's see that  $\|M_\phi\| \geq \|\phi\|_\infty$ . Let  $\varepsilon > 0$  let

$$A_\varepsilon = \{x \in A \mid |\phi(x)| > \|\phi\|_\infty - \varepsilon\} \quad [\text{By abuse of notation, } \phi \text{ is a representative of it eq. (195)}]$$

Then  $A_\varepsilon \in \mathcal{L}(A)$  and  $\lambda(A_\varepsilon) > 0$  by def. of  $\|\phi\|_\infty$ . Find

$$A'_\varepsilon \subset A_\varepsilon \text{ st. } A'_\varepsilon \in \mathcal{L}(A) \text{ and } 0 < \lambda(A'_\varepsilon) < \infty \quad (\text{in assign. 2})$$

$$\text{Let } f = \frac{1}{\lambda(A'_\varepsilon)^{1/p}} \chi_{A'_\varepsilon}$$

$$\text{then } \int_A |f|^p = \int_A \left| \frac{1}{\lambda(A'_\varepsilon)^{1/p}} \chi_{A'_\varepsilon} \right|^p = \int_A \frac{1}{\lambda(A'_\varepsilon)} \chi_{A'_\varepsilon} = \frac{1}{\lambda(A'_\varepsilon)} \int_A \chi_{A'_\varepsilon} = 1$$

$$\Rightarrow \|f\|_p = \left( \int_A |f|^p \right)^{1/p} = (1)^{1/p} = 1 \quad \text{in particular, } f \in L_p(A)$$

$$\begin{aligned} \text{Then } \|M_\phi f\|_p^p &= \int_A |\phi f|^p = \int_A \left| \phi \frac{1}{\lambda(A'_\varepsilon)^{1/p}} \chi_{A'_\varepsilon} \right|^p \\ &= \frac{1}{\lambda(A'_\varepsilon)} \int_A |\phi \chi_{A'_\varepsilon}|^p = \frac{1}{\lambda(A'_\varepsilon)} \int_{A'_\varepsilon} |\phi|^p \geq \frac{1}{\lambda(A'_\varepsilon)} \int_{A'_\varepsilon} (\|\phi\|_\infty - \varepsilon)^p \\ &= (\|\phi\|_\infty - \varepsilon)^p \end{aligned}$$

$$\text{Hence } \|M_\phi f\|_p \geq [(\|\phi\|_\infty - \varepsilon)^p]^{1/p} = \|\phi\|_\infty - \varepsilon$$

$$\text{Now } \|M_\phi\| \geq \|M_\phi f\| \geq \|\phi\|_\infty - \varepsilon$$

$$\|f\|_p \leq 1$$

since  $\varepsilon$  arbitrary, we see  $\|M_\phi\| \geq \|\phi\|_\infty$  □

Theorem: Let  $a < b$  in  $\mathbb{R}$ ,  $f \in L^1[a, b]$

(i) Define  $T_f: L^\infty[a, b] \rightarrow \mathbb{R}$  by  $T_f(\phi) = \int_A f \phi$  then

$T_f$  is a bounded linear functional with  $\|T_f\|_* = \|f\|_1$

(ii) Define  $T_f: C[a, b] \rightarrow \mathbb{R}$  by  $T_f(h) = \int_A f h$  then

$T_f$  is a bounded linear functional with uniform norm

$\|T_f\|_* = \|f\|_1$

(iii) if  $\phi \in L^\infty(A)$ , we have

$$\left| \int_A \phi f \right| \leq \int_A |\phi f| \leq \|\phi\|_\infty \|f\|_1 \quad \text{as seen last class.}$$

It is immediate that  $T_f$  is linear and  $\|T_f\|_* \leq \|f\|_1$

To see that  $\|T_f\|_* \geq \|f\|_1$ ,

let  $\phi = \text{sgn of } f \in L_\infty(A)$

Then  $\|\phi\|_\infty = 1$  (sgn returns only  $\pm 1$  in its range)

and  $\|T_f\|_* \geq |T_f(\phi)| = \left| \int_A f \cdot \underbrace{\text{sgn of } f}_{|f|} \right| = \int_A |f| = \|f\|_1$

(ii) We compute exactly as above that if  $h \in C[a,b] \subset L_\infty[a,b]$  that  $|T_f(h)| = \left| \int_A fh \right| \leq \int_A |fh| \leq \|f\|_1 \|h\|_\infty$  so

$\|T_f\| \leq \|f\|_1$ .

From Assign # 4, Q2 (b) & (c). any measurable simple function, e.g.  $\text{sgn of } f$ , can be approximated pointwise (a.e) by a sequence of continuous functions  $(h_n)_{n=1}^\infty$  st.

$\lim_{n \rightarrow \infty} h_n = \text{sgn of } f$  (a.e) and  $\|h_n\|_\infty = \|\text{sgn of } f\|_\infty = 1$

Then  $|fh_n| = |f||h_n| \leq |f| \|h_n\|_\infty \leq |f|$

so by L.D.C.T.

$|T_f(h_n)| = \left| \int_A fh_n \right| \xrightarrow{n \rightarrow \infty} \left| \int_A f \underbrace{\text{sgn of } f}_{\substack{\uparrow \\ \text{as above}}} \right| = \|f\|_1$

Hence

$\|T_f\|_* \geq \lim_{n \rightarrow \infty} |T_f(h_n)|$  since each  $\|h_n\|_\infty \leq 1$   
 $= \|f\|_1$  □

Theorem If  $1 \leq p < \infty$ ,  $a < b$ , in  $\mathbb{R}$  then

$L_p[a,b]$  is separable.

proof:

Recall that  $C[a,b]$  is separable in the uniform norm.

We note  $\mathbb{Q}[x]$  considered as functions on  $C[a,b]$ , is dense in  $C[a,b]$  We have that  $\mathbb{R}[x]$ , considered as functions on  $C[a,b]$ , is dense in  $C[a,b]$  by Stone-Weierstrass theorem.

by careful approximation  $\mathbb{Q}[x] \xrightarrow{\|\cdot\|_\infty} \mathbb{R}[x]$

If  $q(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{R}[x]$  approximate each  $a_k$  by rational

$r_k$ , and we find  $q(x) = (r_0 + r_1x + \dots + r_nx^n)$   
 $= (a_0 - r_0) + (a_1 - r_1)x + \dots + (a_n - r_n)x^n$

Hence  $\overline{\mathbb{Q}[x]}^{\|\cdot\|_\infty} = C[a,b]$  let  $\mathbb{Q}[x] = \{q_n\}_{n=1}^\infty$   
 uniform closure countable (Exercise)

Now let  $f \in L_p[a,b]$ ,  $\epsilon > 0$ . By Assign. #4, Q2. there is  $h \in C[a,b]$ , st  $\|f-h\|_p < \epsilon/2$ . On the other hand, since

$\mathbb{Q}[x] = \{q_n\}_{n=1}^\infty$  is  $\|\cdot\|_\infty$ -dense in  $C[a,b]$  there is  $n_0 \in \mathbb{N}$

st.  $\|h - q_{n_0}\|_\infty < \frac{\epsilon}{2(b-a)^{1/p}}$  we have

$$\|f - q_{n_0}\|_p \leq \|f - h\|_p + \|h - q_{n_0}\|_p \quad (\text{Minkowski})$$

$$< \epsilon/2 + \left( \int_{[a,b]} |h - q_{n_0}|^p \right)^{1/p} \leq \frac{\epsilon}{2} + \left( \int_{[a,b]} \left| \frac{\epsilon}{2(b-a)^{1/p}} \right|^p \right)^{1/p}$$

$$\leq \left| \frac{\epsilon}{2(b-a)^{1/p}} \right|^p = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \square$$

Note  $L_\infty[a,b]$  is Not Separable.

why? let  $[a,b] = [0,1]$

$$\phi_n = \chi_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}$$

if  $B \subset \mathbb{N}$ .  $\phi_B = \sum_{n \in B} \phi_n$  (pointwise)

check  $(\phi_B)_{B \in \mathcal{P}(\mathbb{N})}$  satisfy

$$\|\phi_B - \phi_{B'}\|_\infty = 1 \quad \text{if } B \neq B'$$

and  $|\{(\phi_B)_{B \in \mathcal{P}(\mathbb{N})}\}| = |\mathcal{P}(\mathbb{N})| = \mathfrak{c}$

Cardinality Use this to show no countable dense subset exists. card. for reals

PMATH 354 Measure Theory and Fourier Analysis Feb 23, 2009

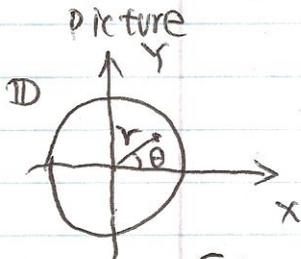
Fourier Analysis

Motivation: Heat equation on a disc

Let  $D$  be a homogeneous disc

it will be convenient to model this as

$$D = \{z \in \mathbb{C} : |z| \leq 1\}$$



polar coordinates

$$x + iy = \underbrace{r}_{\sqrt{x^2+y^2}} (\underbrace{\cos\theta + i\sin\theta}_{e^{i\theta}}) = r e^{i\theta}$$

Given: Temperature on the boundary,  $r=1$ .

$$T(i\theta) = f(\theta), \quad 0 \leq \theta < 2\pi$$

Desire: a function  $u$  defined on  $D$ , which describes the temperature <sup>at  $z$</sup>  given assumption that the temperature is affected only by that on the boundary. This is steady state, not affected by time.

Applied mathematics tells us that

if  $u$  is twice <sup>continuously</sup> partially differentiable on the interior of the disc, then  $u$  is governed by

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \quad (\text{PDE})$$

on  $D^\circ$  (interior)

[in polar coordinates]

$$u(1, \theta) = f(\theta) \quad (\text{BC}) \quad \text{boundary condition}$$

Candidate solutions:  $u_0(r, \theta) = a_0$

$$u_n(r, \theta) = r^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$= r^n \left( \frac{a_n e^{i\theta} + e^{-i\theta}}{2} + b_n \frac{e^{i\theta} - e^{-i\theta}}{2i} \right)$$

$$= r^n (c_n e^{in\theta} + C_n e^{-in\theta})$$

Double check

$$\begin{cases} c_n = \frac{a_n - ib_n}{2} \\ c_{-n} = \frac{a_n + ib_n}{2} \end{cases}$$

Hilroy

General solution to (DE) & (BC)

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$= \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$

(BC)  $\boxed{u(1, \theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = f(\theta)}$  ←

Can we devise a way to test this convergence?

Goal: Figure out how to write

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

Complex valued functions

If  $A \in \mathcal{L}(\mathbb{R})$ , a function  $f: A \rightarrow \mathbb{C}$  is measurable

if  $\text{Re} f, \text{Im} f: A \rightarrow \mathbb{R}$  are both measurable

$f$  is integrable if each

$\text{Re} f, \text{Im} f$  is integrable and

$$\int_A f = \int_A \text{Re} f + i \int_A \text{Im} f$$

Facts (i)  $M_{\mathbb{C}}(A) = \{f: A \rightarrow \mathbb{C} \mid f \text{ is measurable}\}$  is an algebra of functions

have all pointwise operations

(ii) L.D.C.T. holds in this case.

$$\left[ f_n \rightarrow f \text{ pointwise a.e.}, \sum_{n=1}^{\infty} |f_n| \leq g \right]$$

Also, Holder's and Minkowski's modulus non-negative  $\mathbb{R}$ -value inequalities hold.

(iii) M.C.T. and Fatou's Lemma are about non-negative  $\mathbb{R}$ -valued functions.

and can only be applied to such.

From now on, if  $[a, b]$  is a compact interval in  $\mathbb{R}$

we let

$$C[a, b] = \{f: [a, b] \rightarrow \mathbb{C} \mid f \text{ is continuous}\}$$

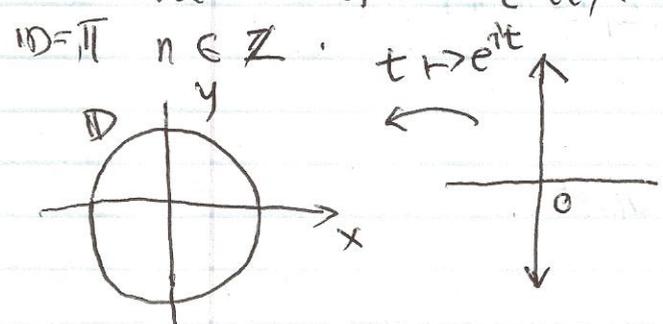
$$L_p[a, b] = \left\{ f: [a, b] \rightarrow \mathbb{C} \mid \begin{array}{l} f \in M_{\mathbb{C}}[a, b] \text{ and} \\ |f|^p \in L_p[a, b] \end{array} \right\} / \sim \text{a.e.}$$

modulus integrable

$L^\infty [a, b] = \{f : [a, b] \rightarrow \mathbb{C} \mid f \in M_\infty [a, b] \text{ and } |f| \text{ is essentially bounded}\}$

Definition Let  $\theta \in \mathbb{R} \setminus \{0\}$ , we say  $f : \mathbb{R} \rightarrow \mathbb{C}$  is (a.e.)  $\theta$ -periodic if  $f(t + \theta) = f(t)$  for (almost every)  $t \in \mathbb{R}$

Note: Define  $e^n(t) := e^{int}$  Then  $e^n$  is  $2\pi$ -periodic.



$\mathbb{R} \ni t \leftrightarrow 2\pi$ -periodic functions on  $\mathbb{R}$   
 $\mathbb{T} \leftrightarrow$  functions on  $\mathbb{T}$

Define  $C(\mathbb{T}) = \{f : \mathbb{R} \rightarrow \mathbb{C} \mid f \text{ is continuous and } 2\pi\text{-periodic}\}$

$$\cong \{f : [-\pi, \pi] \rightarrow \mathbb{C} \mid f \text{ is continuous, } f(-\pi) = f(\pi)\} \subset C[-\pi, \pi]$$

$$1 \leq p < \infty \quad L_p(\mathbb{T}) = \left\{ f : \mathbb{R} \rightarrow \mathbb{C} \mid \begin{array}{l} f \in M_\infty(\mathbb{R}) \\ f \text{ is a.e. } 2\pi\text{-periodic} \\ |f|^p|_{[-\pi, \pi]} \in L[-\pi, \pi] \end{array} \right\}$$

$$\cong L_p[-\pi, \pi]$$

$$L_\infty(\mathbb{T}) = \{f \in \underbrace{L_\infty(\mathbb{R})}_{\substack{\text{measurable} \\ \text{essen. bounded}}} \mid f \text{ is a.e. } 2\pi\text{-periodic}\}$$

Norms  $f \in C(\mathbb{T}), L_\infty(\mathbb{T}) \quad \|f\|_\infty = \text{ess. sup}_{t \in [-\pi, \pi]} |f(t)|$

$$1 \leq p < \infty \quad \|f\|_p = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^p \right)^{1/p} = \frac{1}{(2\pi)^{1/p}} \left( \int_{-\pi}^{\pi} |f|^p \right)^{1/p}$$

clearly, Minkowski's inequality still holds.

Holder holds too:

$$f \in L_p(\mathbb{T}), g \in L_q(\mathbb{T}) \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$\|fg\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |fg| \leq \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} |f|^p \right)^{1/p} \left( \int_{-\pi}^{\pi} |g|^q \right)^{1/q}$$

↑  
Holder =  $\frac{1}{(2\pi)^{1/p} (2\pi)^{1/q}}$

$$= \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^p \right)^{1/p} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |g|^q \right)^{1/q} = \|f\|_p \|g\|_q$$

Feb 25, Li's B-Day 2009

$$C(\mathbb{T}) = \{f: \mathbb{R} \rightarrow \mathbb{C} \mid f \in M_{\mathbb{C}}(\mathbb{R}) \text{ and } \int_{-\pi}^{\pi} |f| < \infty\}$$

$$L_1(\mathbb{T}) = \{f: \mathbb{R} \rightarrow \mathbb{C} \mid f \in M_{\mathbb{C}}(\mathbb{R}) \text{ } f \text{ is } 2\pi\text{-periodic} \} / \sim$$

and  $f|_{[-\pi, \pi]} \in L_1[-\pi, \pi]$

$$\text{Recall } L_1(\mathbb{T}) \supseteq L_p(\mathbb{T}) \supseteq C(\mathbb{T})$$

$$L(\mathbb{T}) \quad (1 < p < \infty \quad \text{continuous})$$

Basic Question: if  $f \in L(\mathbb{T})$ , how may we interpret

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \quad ?$$

what should the  $c_n$ 's look like?

if we allow interchange of  $\sum_{n=-\infty}^{\infty}$  and  $\int_{-\pi}^{\pi}$  with no consideration of convergence, suppose  $f = \sum_{n=-\infty}^{\infty} c_n e^{int}$  ( $e^{in(t)} = e^{int}$ ) and

$f \in L(\mathbb{T})$

$$\int_{-\pi}^{\pi} f(t) e^{ikt} dt = \int_{-\pi}^{\pi} \left( \sum_{n=-\infty}^{\infty} c_n e^{int} \right) e^{-ikt} dt$$

$$\int_{-\pi}^{\pi} f(t) e^{ikt} dt \in L(\mathbb{T}) \quad C(\mathbb{T}) = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} c_n e^{i(n-k)t} dt$$

$$\int_{-\pi}^{\pi} e^{i(n-k)t} dt = \int_{-\pi}^{\pi} \cos((n-k)t) dt + i \int_{-\pi}^{\pi} \sin((n-k)t) dt$$

$$= 0 \quad \text{unless } n=k \quad \underbrace{\int_{-\pi}^{\pi} \sin((n-k)t) dt}_{=0}$$

$$= \begin{cases} 2\pi & n=k \\ 0 & \text{o/w} \end{cases}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{inmkt} dt = c_k 2\pi$$

Definition The  $k$ th Fourier coefficient ( $k \in \mathbb{Z}$ ) of  $f \in L(\mathbb{T})$  is given by

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f e^{-k}$$

Note if  $f=g$  a.e. then  $fe^{-k} = ge^{-k}$  a.e.  $\Rightarrow$   
 $C_k(f) = C_k(g)$

Conclusion.  $C_k(f)$  is well-defined for  $f \in L_1(\mathbb{T})$

Question: Let  $f \in L_1(\mathbb{T}) / L_p(\mathbb{T}) / C(\mathbb{T})$

does  $f = \sum_{n=-\infty}^{+\infty} C_n(f) e^{in} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N C_n(f) e^{in}$  in  $L_1$ -norm /  $L_p$ -norm

Yet another motivating computation:

If  $f \in L_1(\mathbb{T})$ , we let

$S_n(f) = \sum_{k=-n}^n C_k(f) e^{ikt}$  RHS is a cont.  $2\pi$ -periodic func. uniformly (pointwise)

and for  $t \in [-\pi, \pi]$ .  $S_n(f, t) = S_n(f)(t) = \sum_{k=-n}^n C_k(f) e^{ikt}$

We have for  $t \in [-\pi, \pi]$

$S_n(f, t) = \sum_{k=-n}^n C_k(f) e^{ikt} = \sum_{k=-n}^n \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-iks} ds \right) e^{ikt}$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \sum_{k=-n}^n e^{ik(t-s)} ds$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t-s) f(s) ds$   $D_n = \sum_{k=-n}^n e^{ik}$  Dirichlet kernel of order  $n$

$= \frac{1}{2\pi} \int_{-\pi-t}^{\pi-t} D_n(-\sigma) f(\sigma+t) d\sigma$   $\sigma = s-t$  change of variable  $s = \sigma+t$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(-\sigma) f(\sigma+t) d\sigma$

AS  $\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) f(t-s) ds$   $s = -\sigma$

$= D_n * f(t)$  convolution of  $D_n$  with  $f$ .

Convolution:

Let  $X$  be a Banach space

Hilroy

We say  $X$  is homogeneous over  $\mathbb{T}$

if there is a group action  $\mathbb{R} \times X \rightarrow X$   $(t, x) \mapsto t * x$

ie. if  $s, t \in \mathbb{R}$ ,  $x \in X$

- $0 * x = x$
  - $(s+t) * x = s * (t * x)$
  - $s \mapsto s * x : \mathbb{R} \rightarrow X$  cts
  - $(s + 2\pi) * x = s * x$
- } cts action by  $\mathbb{T}$

and moreover we have for  $s \in \mathbb{R}$ ,  $x, y \in X$ ,  $\alpha \in \mathbb{R}$

- $s * (x + y) = s * x + s * y$
  - $s * (\alpha x) = \alpha (s * x)$
  - $\|s * x\| = \|x\|$
- }  $\lambda \mapsto s * x$  is linear  
 $x \mapsto s * x$  preserve distance  
 is isometric

Example: Let  $X = C(\mathbb{T})$   
 continuous  $2\pi$ -periodic functions.  $s \in \mathbb{R}$ .

$f \in C(\mathbb{T})$

$$s * f(t) = f(t-s) \text{ for } t \in \mathbb{R}$$

Then  $C(\mathbb{T})$  with this action, is a homogeneous Banach space over  $\mathbb{T}$ .

Proof: Let  $s, s_1 \in \mathbb{R}$ ,  $f \in C(\mathbb{T})$

We have

- $0 * f(t) = f(t-0) = f(t) \Rightarrow 0 * f = f$
- $(s+s_1) * f(t) = f(t-(s+s_1)) = f((t-s_1)-s) = s * (s_1 * f(t))$
- $2\pi * f(t) = f(t-2\pi) = f(t)$  by  $2\pi$ -periodicity

Easily check  $f, g \in C(\mathbb{T})$ ,  $\alpha \in \mathbb{C}$

$$\text{then for } s \in \mathbb{R}, s * (f+g) = s * f + s * g, s * (\alpha f) = \alpha (s * f)$$

$$\text{Also } \|s * f\|_\infty = \|f\|_\infty$$

Hard part, if  $s \rightarrow s_0$  in  $\mathbb{R}$ , do we have

$$\lim_{s \rightarrow s_0} \|s * f - s_0 * f\|_\infty = 0 \quad ? \quad \text{for fixed } f \in C(\mathbb{T})$$

Recall  $C(\mathbb{T}) \cong \{f: [-\pi, \pi] \rightarrow \mathbb{C} \mid \begin{array}{l} f \text{ is cts.} \\ f(-\pi) = f(\pi) \end{array}\} \subset \mathcal{C}[-\pi, \pi]$

If  $f \in C[-\pi, \pi]$ , since  $[-\pi, \pi]$  is compact,  $f$  is uniformly continuous. Hence given  $\epsilon > 0$ , there is  $\delta > 0$  st.

$$|t - t'| < \delta$$

$\Rightarrow |f(t) - f(t')| < \epsilon$ , thus we have for any  $t \in [-\pi, \pi]$

$$s * f(t) - s_0 * f(t) = f(t-s) - f(t-s_0)$$

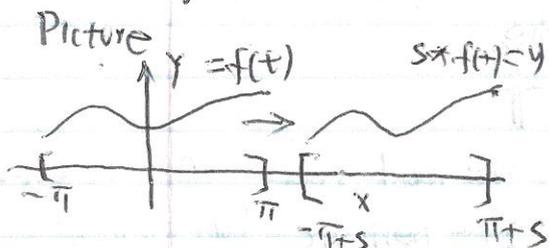
$|t-s - (t-s_0)| = |s_0 - s|$  so if  $|s_0 - s| < \delta$ , we find

$$\|s * f - s_0 * f\|_\infty = \max_{t \in \mathbb{R}} |f(t-s) - f(t-s_0)| < \epsilon \quad \square$$

Feb 27, 2009 PMath 354 Fourier Analysis part

Example Let  $1 \leq p < \infty$ , then

$L_p(\mathbb{T})$  is a homogeneous Banach space over  $\mathbb{T}$ , via the action  $s * f(t) = f(t-s)$  for  $s \in \mathbb{R}$ ,  $f \in L_p(\mathbb{T})$  and a.e.  $t \in \mathbb{R}$ .



proof: It is simple to verify

$$0 * f = f \quad (s+s') * f = s * (s' * f)$$

$$(s+2\pi) * f = s * f$$

$$s * (f+g) = s * f + s * g$$

$$s * (\alpha f) = \alpha (s * f)$$

Let us check that  $\|s * f\|_p = \|f\|_p$ :

$$\|s * f\|_p^p = \frac{1}{2\pi} \int_{-\pi}^{\pi} |s * f|^p = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t-s)|^p dt \stackrel{t=t-s}{=} \frac{1}{2\pi} \int_{-\pi-s}^{\pi-s} |f(t)|^p dt \stackrel{\text{A.S.}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p dt = \|f\|_p^p$$

We also wish to check continuity of  $s \mapsto s * f$  for fixed  $f \in L_p(\mathbb{T})$ .

Given  $\varepsilon > 0$

say  $S \rightarrow S_0$  in  $\mathbb{R}$ . We will find  $\delta > 0$  st.

$$|S - S_0| < \delta \Rightarrow \|S * f - S_0 * f\|_p < \varepsilon$$

First, from A4, find  $h \in C(\mathbb{T})$  st.

$$\|f - h\|_p < \varepsilon/3$$

Then, from the fact that  $C(\mathbb{T})$  is a homogeneous Ban. space over  $\mathbb{T}$ ,

there is  $\delta > 0$  st.  $|S - S_0| < \delta \Rightarrow \|S * h - S_0 * h\|_\infty < \frac{\varepsilon}{3}$

Thus we have for  $|S - S_0| < \delta$

$$\|S * f - S_0 * f\|_p \leq \|S * f - S * h\|_p + \|S * h - S_0 * h\|_p + \|S_0 * h - S_0 * f\|_p$$

$$= \|S * (f - h)\|_p + \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \|S * h - S_0 * h\|_p^p \right)^{1/p} + \|S_0 * (h - f)\|_p$$

$$\leq \|S * h - S_0 * h\|_\infty$$

a.e.

$$\leq \|f - h\|_p +$$

$$\|S * h - S_0 * h\|_\infty + \|h - f\|_p$$

trivial  $< \varepsilon/3 +$

$$\varepsilon/3$$

$$+ \varepsilon/3 = \varepsilon \quad \square$$

Trivial Example:  $X$  any Banach space

for  $S \in \mathbb{R}$ ,  $S * X = X$  for  $x \in X$  then

$X$  is a homogeneous Banach space over  $\mathbb{T}$

Definition of Convolution: Let  $h \in C(\mathbb{T})$  [we could relax this to allow  $h$  to be a piecewise continuous, bounded  $2\pi$ -periodic function]

Let  $X$  be a homogeneous Banach space over  $\mathbb{T}$

Define  $F: \mathbb{R} \rightarrow X$  by  $F(s) = h(s) S * X$

for  $x \in X$

Then  $F$  is a continuous,  $2\pi$ -periodic function from  $\mathbb{R}$  to  $X$

We now define the convolution of  $h$  with  $X$  by

$$h * X := \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{h,x}(s) ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) S * X ds$$

can use any  
interval length  $2\pi$

← vector-valued

Riemann integral

Now suppose  $f \in C(\mathbb{T})$  [or  $f \in L_p(\mathbb{T})$ ] then for (a.e.)  $t \in \mathbb{R}$ .

$$h * f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) s * f(t) ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) f(t-s) ds$$

Notes on convolution:

Given  $h \in C(\mathbb{T})$  [or bdd. piecewise cts  $2\pi$ -periodic]

define  $\Phi_h: X \rightarrow X$  by  $\Phi_h(x) = h * x$

Using properties of vector-valued Riemann integration.

We have that  $\Phi_h$  is a linear operator.

Moreover,  $\Phi_h$  is bounded: if  $x \in X$

$$\begin{aligned} \|\Phi_h(x)\| &= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) s * x ds \right\| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|h(s) s * x\| ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(s)| \|s * x\| ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(s)| \|x\| ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(s)| ds \cdot \|x\| = \|h\|_1 \|x\| \end{aligned}$$

Thus  $\|\Phi_h\| \leq \|h\|_1$  Fore shadowing: If  $X = L_2(\mathbb{T}), C(\mathbb{T})$  then  $\|\Phi_h\| = \|h\|_1$

However, if  $X = L_2(\mathbb{T})$ , sometimes  $\|\Phi_h\| < \|h\|_1$

Approximating with Fourier series  $f \in C(\mathbb{T})$  [ $L_p(\mathbb{T})$ ]

$$S_n(f, t) = \sum_{k=-n}^n C_k(f) e^{ikt} \quad \text{where } C_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-iks} ds$$

if we let  $D_n(t) = \sum_{k=-n}^n e^{ikt}$ , then we have

$$S_n(f, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) f(t-s) ds$$

Computed last class.

Suggests: properties of  $S_n(f) \leftrightarrow$  properties of  $D_n$   
Dirichlet kernel

Theorem (properties of the Dirichlet kernel)

The Dirichlet kernel of order  $n$ ,  $D_n$  satisfies

(i)  $D_n$  is  $\mathbb{R}$ -valued,  $2\pi$ -periodic and even

(ii)  $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n = 1$

(iii) for  $t \in (-\pi, \pi]$

$$D_n(t) = \begin{cases} \frac{\sin[(n+\frac{1}{2})t]}{\sin[\frac{1}{2}t]} & t \neq 0 \\ 2n+1 & t = 0 \end{cases} \quad t \in (-\pi, \pi]$$

(IV)  $\lim_{n \rightarrow \infty} \|D_n\|_1 = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt = \infty$   
 $L_n^1$  Lebesgue constants diverge.

$\|D_n\|_1 = \|D_n\|_1 \quad X = L_1(\pi) \subset C(\pi)$

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Dirichlet kernel,  $D_n(t) = \sum_{k=-n}^n e^{ikt}$

Theorem: (properties of Dirichlet kernel)

proof: (i)  $2\pi$ -periodicity is obvious

evenness, real-valuedness, follow from (iii')

(ii)  $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-n}^n e^{ikt} dt$   
 $= \frac{1}{2\pi} \sum_{k=-n}^n \int_{-\pi}^{\pi} e^{ikt} dt$

Idea:  $\sum_{k=-n}^n r^k \cdot (r^{-\frac{1}{2}} - r^{\frac{1}{2}}) = \text{nice} = \begin{cases} 0 & k \neq 0 \\ 2\pi & k = 0 \end{cases}$

(iii')  $D_n(t) = \sum_{k=-n}^n e^{ikt} = 1 + e^{-it} + \dots + e^{int}$

$$D_n(t) (e^{-i\frac{1}{2}t} + e^{i\frac{1}{2}t}) = \frac{e^{-i(\frac{1}{2}+n)t} + \dots + e^{-i(\frac{1}{2}-1)t} - 1 + e^{i(\frac{1}{2}-1)t} + \dots + e^{i(\frac{1}{2}+n)t}}{e^{-i\frac{1}{2}t} + e^{i\frac{1}{2}t}} = \frac{e^{-i(\frac{1}{2}+n)t} + \dots + e^{-i(\frac{1}{2}-1)t} - [\cos(\frac{1}{2}t) - i\sin(\frac{1}{2}t)] - 1 + [\cos(\frac{1}{2}t) + i\sin(\frac{1}{2}t)] + e^{i(\frac{1}{2}-1)t} + \dots + e^{i(\frac{1}{2}+n)t}}{2i\sin(\frac{1}{2}t)}$$

$$= e^{-i(n+\frac{1}{2})t} - e^{i(n+\frac{1}{2})t}$$

$$= -2i \sin[(n+\frac{1}{2})t]$$

For  $t \in [-\pi, \pi]$   $t \neq 0$

We have  $D_n(t) = \frac{\sin(n+\frac{1}{2})t}{\sin \frac{1}{2}t}$

Also  $D_n(0) = \sum_{k=-n}^n e^{ikt} \Big|_{t=0} = 2n+1$

(IV) Recall  $|\sin \theta| \leq |\theta|$  for  $\theta \in \mathbb{R}$

We have

$$L_n = \|D_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n|$$

$$= \frac{1}{\pi} \int_0^{\pi} |D_n| \quad (\text{evenness})$$

$$= \frac{1}{\pi} \int_0^{\pi} \left| \frac{\sin(n+\frac{1}{2})t}{\sin \frac{1}{2}t} \right| dt \geq \frac{2}{\pi} \int_0^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{t} dt$$

$$\theta = \frac{1}{2}t$$

$$|\sin \frac{1}{2}t| \leq \frac{1}{2}|t|$$

$$\text{tfo } \frac{1}{|\sin \frac{1}{2}t|} \geq \frac{2}{|t|}$$

$$s = (n+\frac{1}{2})t$$

$$t = \frac{s}{(n+\frac{1}{2})}$$

$$= \frac{2}{\pi} \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin s|}{s \cdot \frac{1}{n+\frac{1}{2}}} ds$$

$$\geq \frac{2}{\pi} \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin s|}{s} ds$$

$$= \frac{2}{\pi} \sum_{j=1}^n \int_{(j-1)\pi}^{j\pi} \frac{|\sin s|}{s} ds$$

on  $[(j-1)\pi, j\pi]$  we have  $s \leq j\pi$

$$\geq \frac{2}{\pi} \sum_{j=1}^n \frac{1}{j\pi} \int_{(j-1)\pi}^{j\pi} |\sin s| ds$$

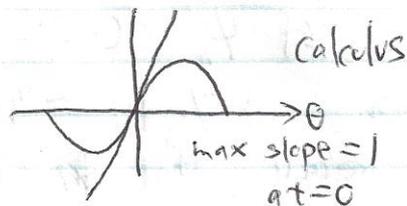
$$= \frac{2}{\pi^2} \sum_{j=1}^n \frac{1}{j} \xrightarrow{n \rightarrow \infty} \infty$$

Hence

$$L_n = \|D_n\|_1 \geq \frac{1}{\pi^2} \sum_{j=1}^n \frac{1}{j} \xrightarrow{n \rightarrow \infty} \infty \quad \square$$

Foreshadowing: We will use the fact that  $\lim_{n \rightarrow \infty} L_n = \infty$  to prove/show that if  $f \in C(\mathbb{T})$ , then

$$S_n(f, t) \xrightarrow{n \rightarrow \infty} f(t) \text{ uniformly in } t$$



Theorem: Let  $h \in C(\mathbb{T})$

(i) If  $\Phi_h : L_1(\mathbb{T}) \rightarrow L_1(\mathbb{T})$  by  $\Phi_h(f) = h * f$

then  $\|\Phi_h\| = \|h\|_1$

(ii) If  $\Psi_h : C(\mathbb{T}) \rightarrow C(\mathbb{T})$  is given by  $\Psi_h(f) = h * f$

then  $\|\Psi_h\| = \|h\|_1$

Proof: Recall that for any convolution operator

$$\Phi_h : X \rightarrow X \quad \Phi_h(x) = h * x$$

hom. Banach space over  $\mathbb{T}$

We have  $\|\Phi_h\| \leq \|h\|_1$

(i) we need only establish that

$$\|\Phi_h\| \geq \|h\|_1$$

Recall  $\|\Phi_h\| = \sup \{\|\Phi_h(f)\|_1 : f \in L_1(\mathbb{T}), \|f\|_1 \leq 1\}$

Define  $f_n = \chi_{[-\frac{1}{n}, \frac{1}{n}]}$  on  $[-\pi, \pi]$  and extend to

a  $2\pi$ -periodic function. Notice

$$\|f_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\pi \chi_{[-\frac{1}{n}, \frac{1}{n}]}|$$

$$= \frac{n}{2} \int_{-\pi}^{\pi} \chi_{[-\frac{1}{n}, \frac{1}{n}]} = \frac{n}{2} \cdot \frac{2}{n} = 1$$

Now we have  $(h * f_n)(t) = \frac{1}{n} \int_{-\pi}^{\pi} h(s) f_n(t-s) ds$

always everywhere as  $t \in t + [-\pi, \pi]$

$$= \frac{n}{2} \int_{-\pi}^{\pi} h(s) \chi_{[-\frac{1}{n}, \frac{1}{n}]}(t-s) ds$$

$$= \frac{n}{2} \int_{t-\frac{1}{n}}^{t+\frac{1}{n}} h(s) \chi_{[-\frac{1}{n}, \frac{1}{n}]}(s) ds =$$

$$\frac{n}{2} \int h(st+t) \chi_{[-\frac{1}{n}, \frac{1}{n}]}(s) ds$$

$$= \frac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} h(st+t) \chi_{[-\frac{1}{n}, \frac{1}{n}]}(s) ds$$

$$= \frac{h}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} n(st+t) ds$$

Now  $\|h - h * f_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |h - h * f_n|$   
 $= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| h(t) - \frac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} h(s) ds \right| dt$   
 $= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} (h(t) - h(t+s)) ds \right| dt$   
 $\leq \frac{n}{2\pi} \int_{-\pi}^{\pi} \int_{-\frac{1}{n}}^{\frac{1}{n}} |h(t) - h(t+s)| ds dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\frac{1}{n}}^{\frac{1}{n}} \sup_{s \in (-\frac{1}{n}, \frac{1}{n})} |h(t) - (t-s) * h(t)| ds dt$   
 $\leq \sup_{s \in (-\frac{1}{n}, \frac{1}{n})} |h(t) - (t-s) * h(t)| \leq \sup_{s \in (-\frac{1}{n}, \frac{1}{n})} |h - (t-s) * h| \xrightarrow{n \rightarrow \infty} 0$

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Theorem: (i) There is a set  $U \subset L_1(\mathbb{T})$  whose complement is meager st.  $\sup_{n \in \mathbb{N}} \|S_n(f)\|_1 = \infty$  for  $f \in U$ . In particular  $\lim_{n \rightarrow \infty} S_n(f) \neq f$  for  $f \in U$

(ii) There is a subset  $U \subset C(\mathbb{T})$  whose complement is meager st.  $\sup_{n \in \mathbb{N}} \|S_n(h)\|_{\infty} = \infty$  for  $h \in U$ .

Proof: (i) Recall  $S_n(f) = \sum_{j=-n}^n c_j(f) e^{ijt} = D_n * f$   
 $\frac{1}{2\pi} \int_{-\pi}^{\pi} f e^{ijt} e^{i(t-s)} = e^{ijt}$

We have  $\|D_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n| = L_n \xrightarrow{n \rightarrow \infty} \infty$   
Lebesgue const.

Also  $\sup \{ \|S_n(f)\|_1 : f \in L_1(\mathbb{T}), \|f\|_1 \leq 1 \} = \sup \{ \|D_n * f\|_1, \dots \}$

$= \|D_n\|_1 = L_n = \infty$   
Thm

$D_n * f = D_n * f$   
 Convolution operator

By (Corollary to) the Banach-Steinhaus Theorem

$$F = \left\{ f \in L_1(\mathbb{T}) \mid \sup_{n \in \mathbb{N}} \underbrace{\| \Phi_n(f) \|_1}_{\| S_n(f) \|_1} < \infty \right\}$$

Then  $F$  is meager (i.e. of 1st category) in  $L_1(\mathbb{T})$

Barre Category thm  $\Rightarrow L_1(\mathbb{T})$  is non-meager  $\Rightarrow U = L_1(\mathbb{T}) \setminus F \neq \emptyset$

(ii) Similar  $\square$

Two ways to go from here  $\left\{ \begin{array}{l} \text{averaging Fourier sums (Fejer)} \\ \text{look at specific functions Localization} \\ \text{Dini's Thm} \end{array} \right.$

Averaging to the rescue.

Definition: Let  $X$  be a Banach space  $\{x_n\}_{n=1}^{\infty} \subset X$   
 Define the  $n$ th Cesaro average/mean of  $\{x_n\}_{n=1}^n$  by

$$\sigma_n(x) = \frac{1}{n} (x_1 + \dots + x_n)$$

Proposition: If  $\lim_{n \rightarrow \infty} x_n$  exists, then  $\lim_{n \rightarrow \infty} \sigma_n(x)$  exists  
 and is equal to  $\lim_{n \rightarrow \infty} x_n$  (Proof: Exercise)

If  $f \in L(\mathbb{T})$  [or  $f \in L_1(\mathbb{T})$ ]

We define  $\sigma_n(f) = \frac{1}{n+1} \sum_{j=0}^n S_j(f) = \frac{1}{n+1} \sum_{j=0}^n \sum_{k=-j}^j c_k(f) e^{ik}$

$$= \frac{1}{n+1} \sum_{j=0}^n D_j * f = \left( \frac{1}{n+1} \sum_{j=0}^n D_j \right) * f$$

We let  $K_n = \frac{1}{n+1} \sum_{j=0}^n D_j$  and

$$= \frac{1}{n+1} \sum_{j=0}^n \sum_{k=-j}^j e^{ik} \quad \text{and call this the Fejer}$$

kernel of order  $n$ .

Philosophy behaviour of  $K_n \iff$  convergence of  $\sigma_n(f)$

Theorem: (Properties of Fejer kernel)

The Fejer kernel of order  $n$ ,  $K_n$

satisfies (i)  $K_n$  is real valued, continuous and  $2\pi$ -periodic

$$(ii) K_n(t) = \begin{cases} \frac{1}{n+1} \left( \frac{\sin \frac{1}{2}(n+1)t}{\sin \frac{1}{2}t} \right)^2 & t \neq 0 \\ n+1 & t = 0 \end{cases} \text{ for } t \in [-\pi, \pi]$$

$$(iii) \|K_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_n| = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n = 1$$

(iv) Zf  $0 < |t| \leq \pi$  then

$$0 \leq K_n(t) \leq \frac{\pi^2}{(n+1)t^2}$$

Proof (i) since each  $D_j$   $j=0, 1, \dots, n$  is  $\mathbb{R}$ -valued, continuous and

$2\pi$ -periodic, the same is true for  $K_n = \frac{1}{n+1} \sum_{j=0}^n D_j$

$$(ii) K_n(t) = \frac{1}{n+1} \sum_{j=0}^n \sum_{k=-j}^j e^{ikt} = \frac{1}{n+1} \left[ \sum_{k=-n}^n (n+1-|k|) e^{ikt} \right]$$

$$= \frac{1}{n+1} [ e^{-int} + 2e^{-i(n-1)t} + \dots + ne^{-it} + (n+1) + ne^{it} + \dots + e^{int} ]$$

Now consider

$$(n+1) K_n(t) (e^{-it} - 2 + e^{it})$$

$$= e^{-i(n+1)t} + 2e^{-i(n)t} + 3e^{-i(n-1)t} + \dots + (n+1)e^{-it} + n + (n-1)e^{it} + \dots + e^{i(n-1)t}$$

$$- (2e^{-int} + 2 \cdot 2 \cdot e^{-i(n)t} + \dots + 2ne^{-it} + 2(n+1) + 2ne^{it} + 2e^{i(n)t} + \dots)$$

$$+ e^{-i(n+1)t} + \dots + (n-1)e^{-it} + (n+1) + (n+1)e^{it} + \dots$$

$$+ (n-2)e^{i(n-1)t} + ne^{it} + e^{i(n+1)t}$$

$$- e^{-i(n-1)t} - 2 + e^{i(n-1)t} \text{ if } t \in (-\pi, \pi) \setminus \{0\}. \text{ Then}$$

$$k_n(t) = \frac{1}{n+1} \frac{e^{-i(n+1)t} - 2 + e^{i(n+1)t}}{e^{it} - 2te^{it}} = \frac{(e^{i(\frac{n+1}{2}t)} - e^{-i(\frac{n+1}{2}t)})^2}{(e^{it/2} - e^{-it/2})^2}$$

$$= \frac{1}{n+1} \frac{(-i \sin \frac{1}{2}(n+1)t)^2}{(-i \sin \frac{1}{2}t)^2} = \left| \frac{\sin \frac{1}{2}(n+1)t}{\sin \frac{1}{2}t} \right|^2$$

$$\text{Also } k_n(0) = \frac{1}{n+1} \sum_{k=0}^n e^{ik \cdot 0} = \frac{1}{n+1} \sum_{k=0}^n (2)^{k+1}$$

$$= \frac{1}{n+1} \left[ 2 \frac{2^{n+1} - 1}{2 - 1} \right] = n+1$$

(iii) since  $k_n \geq 0$  on  $[-\pi, \pi]$

$$\|k_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n| = \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n = \frac{1}{2\pi} \frac{1}{n+1} \sum_{k=0}^n \int_{-\pi}^{\pi} e^{ikt} = 1$$

$= 0$  unless  $k=0$

(iv) we have  $\frac{2}{\pi} (\frac{2}{\pi} \leq \sin t)$  for  $0 \leq t \leq \frac{\pi}{2}$

so for choice of  $0 \leq t \leq \pi$

$$\frac{1}{\sin \frac{1}{2}t} \leq \frac{1}{t/\pi} \quad \frac{1}{\sin \frac{1}{2}t} \leq \frac{1}{t/\pi} \leq \frac{2\pi}{t}$$

$$k_n(t) = \frac{1}{n+1} \left( \frac{\sin \frac{1}{2}(n+1)t}{\sin \frac{1}{2}t} \right)^2 \leq \frac{1}{(n+1) \left(\frac{t}{\pi}\right)^2} \leq \frac{1}{(n+1)^2 \left(\frac{\pi}{t}\right)^2} = \frac{t^2}{(n+1)^2 \pi^2}$$

for  $t > 0$ , this also holds for  $t < 0$  since  $k_n$  is even:

$$k_n(-t) = k_n(t)$$

Mar 9, 2009 Pmath 354 Measure Theory & Fourier Analysis

Definition: A sequence  $\{k_n\}_{n=1}^{\infty}$  of (piecewise) continuous,  $\mathbb{R}$ -valued,  $2\pi$ -periodic functions on  $\mathbb{R}$  is a summability kernel provided

$$(i) \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n = 1 \quad \text{limit to average 1}$$

$$(ii) \sup_{n \in \mathbb{N}} \|k_n\|_1 = \sup_{n \in \mathbb{N}} \frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n| < \infty \quad \text{"L-bounded mass"}$$

(iii) If  $0 < \delta \in \pi$

$$\lim_{n \rightarrow \infty} \left( \int_{-\delta}^{\delta} |k_n| + \int_{\delta}^{\pi} |k_n| \right) = 0 \quad \text{"mass concentrates" about 0}$$

limit

Examples (i) Fejer kernel  $\{K_n\}_{n=1}^{\infty}$  is a summability kernel.

see that (i) (ii) are satisfied

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n = \|K_n\|_1 = 1 \xrightarrow{n \rightarrow \infty} 1 < \infty$$

(ii) The Dirichlet kernel  $\{D_n\}_{n=1}^{\infty}$  is NOT a summability kernel.

see that condition (ii) fails.

$$L_n = \|D_n\|_1 \xrightarrow{n \rightarrow \infty} \infty$$

(iii)  $k_n = n\pi \chi_{[-\frac{1}{n}, \frac{1}{n}]}$  on  $[-\pi, \pi]$  continued

$2\pi$ -periodically on  $\mathbb{R}$ .

check (i) (ii) (iii) all trivial

(iv)  $g_n(t) = C_n e^{-nt^2}$   $t \in [-\pi, \pi]$  ctd  $2\pi$ -periodically

$\uparrow$  normalisation const so  $\|g_n\|_1 = 1$

Check with  $C_n$  chosen appropriately (i) (ii) are fine (iii) use "calculus estimates"

(iv)  $f_n(x) = C_n \frac{\sin(nx)}{x}$   $x \in [-\pi, \pi] \setminus \{0\}$  ctd  $2\pi$ -periodically  
 $(= \frac{1}{n}, x=0)$   
 $\uparrow$  normalisation const.

Choose to satisfy (i)

Note: Visit the course website for graphs of  $D_{10}$  and  $K_{10}$ .

Abstract summability kernel theorem.

Let  $X$  be a homogeneous Banach space over  $\mathbb{T}$

If  $\{K_n\}_{n=1}^{\infty}$  is a summability kernel, then

$$\lim_{n \rightarrow \infty} \|k_n * X - X\| = 0$$

← norm in X

proof: Fix  $x \in X \setminus \{0\}$  Define  $F: \mathbb{R} \rightarrow X$  by

$$F(s) = S * X$$

$$\text{with } F(0) = 0 * X = X$$

So  $F$  is continuous and  $2\pi$ -periodic. Then by definition

$$k_n * X = \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(s) \underbrace{S * X}_{F(s)} ds$$

Thus we have

$$\|k_n * X - X\| = \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(s) F(s) ds - F(0) \right\|$$

$$= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(s) F(s) ds - \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(s) F(0) ds \right\|$$

$$= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(s) (F(s) - F(0)) ds \right\| \quad \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n = 1 \text{ by axiom (i')} \right)$$

$$\stackrel{A1}{\leq} \frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n(s)| \|F(s) - F(0)\| ds \quad (ff-)$$

Pick  $0 < \delta \leq \pi$

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n(s)| \|F(s) - F(0)\| ds &= \frac{1}{2\pi} \int_{-\pi}^{-\delta} |k_n(s)| \|F(s) - F(0)\| ds \\ &+ \underbrace{\frac{1}{2\pi} \int_{-\delta}^{\delta} |k_n(s)| \|F(s) - F(0)\| ds}_{(f)} + \underbrace{\frac{1}{2\pi} \int_{\delta}^{\pi} |k_n(s)| \|F(s) - F(0)\| ds}_{(ff+)} \end{aligned}$$

Let's look at (ff-)

$$\begin{aligned} \int_{-\pi}^{-\delta} |k_n(s)| \|F(s) - F(0)\| ds &\leq \int_{-\pi}^{-\delta} |k_n(s)| \left[ \sup_{t \in (-\pi, \pi)} \|F(t)\| \right] ds \\ &\leq \|F(s)\| + \|F(0)\| \\ &\leq \sup_{t \in (-\pi, \pi)} 2\|F(t)\| \end{aligned}$$

$\|t * X\| = \|X\|$   
by axiom of Banach space

$$= \int_{-\pi}^{-\delta} |k_n(s)| 2\|X\| ds = 2\|X\| \int_{-\pi}^{-\delta} |k_n(s)| ds \xrightarrow{n \rightarrow \infty} 0$$

axiom (B) of  $\{k_n\}_{n=1}^{\infty}$

similarly:  $(f f +) \xrightarrow{n \rightarrow \infty} 0$  too.

Let's examine  $(f)$

$$\frac{1}{2\pi} \int_{-\delta}^{\delta} |k_n(s)| \|F(s) - F(0)\| ds \leq \frac{1}{2\pi} \int_{-\delta}^{\delta} |k_n(s)| \left[ \sup_{t \in [-\delta, \delta]} \|F(s) - F(0)\| \right] ds$$

$$\leq \sup_{t \in [-\delta, \delta]} \|F(t) - F(0)\|$$

constant term of  $S$   
depends only on  $\delta$

$$= \left[ \sup_{t \in [-\delta, \delta]} \|F(t) - F(0)\| \right] \frac{1}{2\pi} \int_{-\delta}^{\delta} |k_n(s)| ds$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n(s)| ds \leq C :=$$

$$\leq \sup_{t \in [-\delta, \delta]} \|F(t) - F(0)\| \cdot C \xrightarrow[\text{F iscts}]{\delta > 0} 0$$

$\sup_{n \in \mathbb{N}} \frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n| < \infty$

Thus for any  $\delta > 0$  we have

$$\limsup_{n \rightarrow \infty} \|k_n * x - x\| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n(s)| \|F(s) - F(0)\| ds$$

$$\xrightarrow{n \rightarrow \infty} \leq \sup_{t \in [-\delta, \delta]} \|F(t) - F(0)\| \cdot C$$

Since  $\sup_{t \in [-\delta, \delta]} \|F(t) - F(0)\| \xrightarrow{\delta > 0} 0$  we find

$$0 \leq \limsup_{n \rightarrow \infty} \|k_n * x - x\| \leq 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \|k_n * x - x\| = 0 \quad \square$$

Corollary: (i) if  $f \in C(\mathbb{T})$  then  $\lim_{n \rightarrow \infty} \|O_n(f) - f\|_{\infty} = 0$

(ii) if  $f \in L_p(\mathbb{T})$   $1 \leq p < \infty$ , Then

$$\lim_{n \rightarrow \infty} \|O_n(f) - f\|_p = 0$$

proof: (i) We saw that for  $f \in C(\mathbb{T})$

$\sigma_n(f) = k_n * f$  where  $\{k_n\}_{n=1}^{\infty}$  is the Fejor kernel.

[ Note axioms (i) (i') are trivial, to see (ii) of summability kernel

$$\text{If } 0 < \delta \leq \pi, \quad k_n(t) \leq \frac{\pi^2}{(n+1)t^2}$$

By even symmetry of  $k_n = |k_n|$

$$0 \leq \int_{-\pi}^{-\delta} k_n + \int_{\delta}^{\pi} k_n = 2 \int_{\delta}^{\pi} k_n(t) dt \leq \frac{2\pi^2}{n+1} \int_{\delta}^{\pi} \frac{1}{t^2} dt$$

By Abstract summability kernel thm.  $\leq \frac{\pi^2}{(n+1)t^2} = \frac{2\pi^2}{n+1} \left( \frac{1}{\delta} - \frac{1}{\pi} \right) \xrightarrow{n \rightarrow \infty} 0$

$$\lim_{n \rightarrow \infty} \|k_n * f - f\|_{\infty} = 0$$

proof of (ii) is nearly identical  $\square$

Mar 11, 2009 PMath 354. Measure Theory and Fourier Analysis.

Last time

$f \in L_1(\mathbb{T})$

$$\sigma_n(f) = \frac{1}{n+1} \sum_{j=0}^n S_j(f) = \frac{1}{n+1} \sum_{k=0}^n \sum_{k=j}^j c_k(f) e^{ikt} \stackrel{\text{y1}}{\sim} e^{ikt} = e^{ikt}$$

We saw

$$\| \cdot \|_1 - \lim_{n \rightarrow \infty} \sigma_n(f) = f \text{ i.e.}$$

$$\lim_{n \rightarrow \infty} \| \sigma_n(f) - f \|_1 = 0$$

We recall that  $f = g$  a.e. i.e.  $f = g$  in  $L_1(\mathbb{T})$

$$\Rightarrow c_k(f) = c_k(g) \text{ for all } k \in \mathbb{Z}$$

Corollary: If  $f, g \in L_1(\mathbb{T})$  and  $c_k(f) = c_k(g)$  then  $f = g$  a.e.  $f = g$  in  $L_1(\mathbb{T})$

proof: If  $c_k(f) = c_k(g)$  for every  $k \in \mathbb{Z}$ , then

$$\sigma_n(f) = \sigma_n(g) \text{ for each } n = 0, 1, 2, \dots$$

Hence

$$f = \| \cdot \|_1 - \lim_{n \rightarrow \infty} \sigma_n(f) = \| \cdot \|_1 - \lim_{n \rightarrow \infty} \sigma_n(g) = g \quad \boxed{11}$$

↑  
from assumptions

A more classical setting:

$$\text{Let } L(\mathbb{T}) = \left\{ f: \mathbb{R} \rightarrow \mathbb{C} \mid \begin{array}{l} f \text{ is a.e. } 2\pi\text{-periodic.} \\ f \text{ is measurable} \\ \text{and } \int_{-\pi}^{\pi} |f| < \infty \end{array} \right\}$$

[Note.  $L_1(\mathbb{T}) = L(\mathbb{T}) / \sim_{\text{a.e}}$ ]

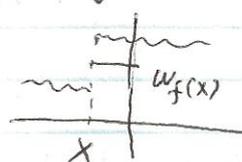
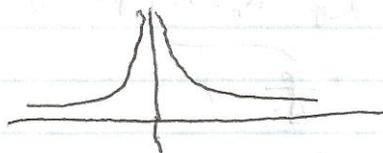
if  $f \in L(\mathbb{T})$   $x \in \mathbb{R}$  (usually  $x \in [-\pi, \pi]$ ) we define the mean-value of  $f$  at  $x$  by

$$w_f(x) = \lim_{\delta \rightarrow 0} \frac{1}{2\delta} [f(x+\delta) + f(x-\delta)]$$

provided the limit exists. If  $f$  is  $\mathbb{R}$ -valued,  $\pm \infty$  are acceptable values for the limit.

E.g.  $f(t) = \frac{1}{\sqrt{|t|}}$  for  $t \in [-\pi, \pi]$  extended  $2\pi$ -periodically to  $\mathbb{R}$ .

$$w_f(0) = +\infty$$



**Fejér's Theorem:** (i) if  $f \in L(\mathbb{T})$ ,  $x \in [-\pi, \pi]$

and  $w_f(x)$  exists. then  $\lim_{n \rightarrow \infty} \sigma_n(f, x) = \lim_{n \rightarrow \infty} \sigma_n(f)(x) = w_f(x)$

(ii) If  $[a, b]$  is a closed subinterval of  $[-\pi, \pi]$  on which  $f$  is continuous. then

$$\lim_{n \rightarrow \infty} \sigma_n(f, x) = f(x) \text{ uniformly for } x \in [a, b]$$

$$\sup_{x \in [a, b]} |\sigma_n(f, x) - f(x)| \xrightarrow{n \rightarrow \infty} 0$$

proof (i) Recall some properties of the Fejér kernel  $K_n$

$$(i) \underbrace{K_n * f(x)}_{\text{in sense of A5}} = \sigma_n(f, x)$$

(ii)  $K_n$  is even, non-negative  $2\pi$ -periodic

$$(iii) \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_n| = 1$$

(iv) If  $0 < |t| \leq \pi$  then  $k_n(t) \leq \frac{\pi^2}{(n+1)t^2}$

In particular, if  $\delta > 0$ , then  $k_n(t) \leq \frac{\pi^2}{(n+1)\delta^2}$  for  $t \in [\delta, \pi]$  or for  $t \in [-\pi, -\delta]$

Now suppose  $x \in [-\pi, \pi]$  and

$$w_f(x) = \lim_{s \rightarrow 0} \frac{1}{2} [f(x+s) + f(x-s)] \text{ exists.}$$

$|w_f(x)| < \infty$  [The case  $w_f(x) = \infty$  is on A5]

Let  $\varepsilon > 0$  be given > choose  $\delta < \pi$  such that

$$0 < |s| \leq \delta \quad \left| \frac{1}{2} [f(x+s) + f(x-s)] - w_f(x) \right| < \varepsilon$$

we have

$$\left| \sigma_n(f, x) - w_f(x) \right| = \left| k_n * f(x) - w_f(x) \right|$$

$$= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(s) f(x-s) ds - w_f(x) \right|$$

Riemann integral      Lebesgue integral

$$= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(s) (f(x-s) - w_f(x)) ds \right| \quad (f)$$

$$\leq \frac{1}{2\pi} \underbrace{\int_{-\pi}^{-\delta} k_n(s) |f(x-s) - w_f(x)| ds}_{(ff-)} + \frac{1}{2\pi} \int_{-\delta}^{\delta} k_n(s) |f(x-s) - w_f(x)| ds + \frac{1}{2\pi} \underbrace{\int_{\delta}^{\pi} k_n(s) |f(x-s) - w_f(x)| ds}_{(ff+)}$$

First let's look at (ff+) we have

$$k_n(s) \leq \frac{\pi^2}{(n+1)\delta^2} \quad \text{for } s \in [\delta, \pi]$$

$$\text{so } (ff+) \leq \frac{1}{2\pi} \int_{\delta}^{\pi} \frac{\pi^2}{(n+1)\delta^2} \cdot |f(x-s) - w_f(x)| ds$$

$$= \frac{\pi}{2(n+1)\delta^2} \int_{\delta}^{\pi} |x * f^{\vee} - w_f(x)| ds \quad \text{as } n \rightarrow \infty \rightarrow 0$$

$$\leq \frac{\pi}{2(n+1)\delta^2} \int_{-\pi}^{\pi} |x * f^{\vee} - w_f(x)| ds = \frac{\pi^2}{(n+1)t^2} \|x * f^{\vee} - w_f(x)\|_1$$

Integrable const. func.

similarly,  $(ff^-) \xrightarrow{n \rightarrow \infty} 0$  Now let's check  $(f)$

Note

$$\int_{-h}^h K_n(s) (f(x-s) - W_f(x)) ds$$

$$s \rightarrow -s = \int_{-h}^h K_n(s) (f(x+s) - W_f(x)) (-1) ds = \int_{-h}^h K_n(s) (f(x+s) - W_f(x)) ds$$

Thus

$$\frac{1}{2h} \int_{-h}^h K_n(s) (f(x-s) - W_f(x)) ds$$

some Lebesgue integral extends Riemann.

$$= \frac{1}{2h} \left[ \int_{-h}^h K_n(s) (f(x-s) - W_f(x)) ds + \int_{-h}^h K_n(s) (f(x+s) - W_f(x)) ds \right]$$

$$= \frac{1}{2h} \int_{-h}^h K_n(s) \left[ \frac{1}{2} (f(x-s) + f(x+s)) - W_f(x) \right] ds$$

Hence

$$\left| \frac{1}{2h} \int_{-h}^h K_n(s) \left[ \frac{1}{2} (f(x+s) - f(x-s)) - W_f(x) \right] ds \right|$$

$$\leq \frac{1}{2h} \int_{-h}^h K_n(s) \left| \frac{1}{2} (f(x+s) + f(x-s)) - W_f(x) \right| ds$$

by choice of  $h$

$$\leq \frac{1}{2h} \int_{-h}^h K_n(s) \epsilon ds \leq \frac{4}{2h} \int_{-h}^h K_n(s) ds = \epsilon$$

Hence  $\limsup_{n \rightarrow \infty} \left| \frac{1}{2h} \int_{-h}^h K_n(s) [f(x+s) - W_f(x)] \right| \leq \epsilon$

Thus we find

$$0 \leq \limsup_{n \rightarrow \infty} |O_n(f, x) - W_f(x)| \leq \epsilon$$

Since  $\epsilon$  is chosen arbitrarily, we find

$$\limsup_{n \rightarrow \infty} |O_n(f, x) - W_f(x)| = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} |O_n(f, x) - W_f(x)| = 0$$

(ii) Note that if  $f$  is cts.  $[a, b]$ , then  $f$  is uniformly cts on  $[a, b]$  thus, given  $\epsilon > 0$ , as in (i) above, the  $h$  can be chosen uniformly for all choices of  $x \in [a, b]$ . Hence all approximations above can be done uniformly in such  $x$ .  $\square$

Mar 6, 2009

$\epsilon > 0$   $\epsilon$ -embedding Metric embedding  
 $\mathbb{R}^{O(\epsilon^{-2} \log n)}$  Johnson-Lindenstrauss 84

Graph Realization. Rank minimization

Sensor localization

Mar 13, 2009 PMath 354 Measure Theory and Fourier Analysis

Erratum A5 Q4

$$\liminf_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(s) f(x-s) ds \geq M$$

optional                      important

Last time = Fejers Theorem

$f \in L(\mathbb{T})$   $x \in \mathbb{R}$ ,  $w_f(x) = \lim_{s \rightarrow 0} \frac{1}{2} (f(x/s) + f(x-s))$  exists

then  $\sigma_n(f, x) = \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(s) f(x-s) ds}_{\text{Lebesgue integral A5}} \xrightarrow{n \rightarrow \infty} w_f(x)$

proposition: Let  $f \in L(\mathbb{T})$   $x \in \mathbb{R}$

zf  $S_n(f, x) = \sum_{j=-n}^n c_j(f) e^{ijx}$  converges as  $n \rightarrow \infty$

then  $\lim_{n \rightarrow \infty} S_n(f, x) = \lim_{n \rightarrow \infty} \sigma_n(f, x)$

proof: zf a sequence of real (complex numbers) converges, so to does the sequence of its Cesaro means, and converges to the same number.

Corollary (to Fejer) zf  $f \in L(\mathbb{T})$   $x \in \mathbb{R}$  st.

$w_f(x)$  exists and  $\lim_{n \rightarrow \infty} S_n(f, x)$  converges.

then  $\lim_{n \rightarrow \infty} S_n(f, x) = w_f(x)$

Improvements to Fejer's Theorem

Definition: Let  $f \in L[a, b]$ , and  $x \in (a, b)$

We say  $x$  is a Lebesgue point of  $f$  if

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h \left| \frac{1}{2} (f(x-s) + f(x+s)) - f(x) \right| ds \text{ exists}$$

PMath 454 fact:

Almost every  $x \in (a, b)$  is a Lebesgue point for  $f$ .

Lebesgue - Fejer Theorem:

Zf  $f \in L(\mathbb{T})$ ,  $x \in [-\pi, \pi]$  is a Lebesgue point for  $f$ , then

$$(f) \lim_{n \rightarrow \infty} \sigma_n(f, x) = f(x)$$

Hence (f) holds for a.e  $x \in [-\pi, \pi]$

On the coefficients of Fourier series

Question: Zf  $(c_n)_{n \in \mathbb{Z}}$  is a sequence of real (complex) numbers, is there  $f \in L_1(\mathbb{T})$  s.t.

$$c_n = c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \text{ for } n \in \mathbb{Z}?$$

Lemma: Zf  $f \in L_1(\mathbb{T})$  and  $n \in \mathbb{Z}$  then

$$|c_n(f)| \leq \|f\|_1$$

$$\begin{aligned} \text{proof: } |c_n(f)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| \underbrace{|e^{-int}|}_{=1} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f| = \|f\|_1 \quad \square \end{aligned}$$

Riemann - Lebesgue Lemma: Zf  $f \in L_1(\mathbb{T})$ , then

$$\lim_{|n| \rightarrow \infty} c_n(f) = 0$$

proof: From (a corollary to) the Abstract - Summability Theorem

$$\lim_{n \rightarrow \infty} \|\sigma_n(f) - f\|_1 = 0$$

Thus given  $\varepsilon > 0$ , there is  $n_0 \in \mathbb{N}$  s.t.

$$\|\sigma_n(f) - f\|_1 < \varepsilon \text{ for } n \geq n_0$$

Hilroy

Thus if  $|k| > n_0$ , we have

$$C_k(\sigma_{n_0}(f) - f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sigma_{n_0}(f) - f) e^{-ik} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{j=-n_0}^{n_0} b_j e^{ij} - f \right) e^{-ik}$$

finite sum

$$\in \text{span} \{ e^{-n_0}, e^{-(n_0+1)}, \dots, e^{n_0}, e^{n_0+1} \} = \frac{1}{2\pi} \sum_{j=-n_0}^{n_0} b_j \int_{-\pi}^{\pi} e^{i(j-k)t} dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f e^{-ik}$$

$e^{ij(t)} := e^{ijt}$

$$\sigma_{n_0}(f) = \sum_{j=-n_0}^{n_0} b_j e^{ij} \quad \text{--- defn. of span} \quad = 0 - C_k(f)$$

$\forall \subseteq \mathbb{C}$

Thus for  $|k| > n_0$

lemma

$$|C_k(f)| = |C_k(\sigma_{n_0}(f) - f)| \leq \| \sigma_{n_0}(f) - f \|_1 < \epsilon$$

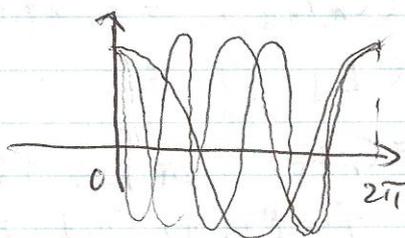
Thus  $\lim_{|k| \rightarrow \infty} C_k(f) = 0$

□  
- almost everywhere ae. class

Corollary: If  $f \in L_1(\mathbb{T})$  [ $f \in L(\mathbb{T})$ ]

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(t) \cos(nt) dt = 0$$

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(t) \sin(nt) dt = 0$$



Proof:  $\cos(nt) = \frac{1}{2}(e^{int} + e^{-int})$

So  $\int_{-\pi}^{\pi} f(t) \cos(nt) dt = \frac{1}{2} \left[ \int_{-\pi}^{\pi} f(t) e^{int} dt + \int_{-\pi}^{\pi} f(t) e^{-int} dt \right]$

$$= \pi (C_{-n}(f) + C_n(f)) \xrightarrow[n \rightarrow \infty]{\text{R-L Lemma}} 0$$

The "sin" limit is similar

Let  $A(\mathbb{Z}) = \{ (C_n(f))_{n \in \mathbb{Z}} : f \in L_1(\mathbb{T}) \}$

Riemann-Lebesgue Lemma  $\Rightarrow$

$$A(\mathbb{Z}) = C_0(\mathbb{Z}) = \{ (C_n)_{n \in \mathbb{Z}} : \lim_{|n| \rightarrow \infty} C_n = 0 \}$$

vector space under pointwise operations

$$(C_n)_{n \in \mathbb{Z}} + (d_n)_{n \in \mathbb{Z}} = (C_n + d_n)_{n \in \mathbb{Z}}$$

$$\alpha (C_n)_{n \in \mathbb{Z}} = (\alpha C_n)_{n \in \mathbb{Z}}$$

$C_0(\mathbb{Z})$  is a Banach space with norm  $\| (c_n)_{n \in \mathbb{Z}} \|_\infty = \sup_{n \in \mathbb{Z}} |c_n|$

Question: Is the map  $f \mapsto (c_n)_{n \in \mathbb{Z}} : L_1(\mathbb{T}) \rightarrow C_0(\mathbb{Z})$  surjective?

Answer: NO!

PM 453 Theorem: (Open Mapping Theorem)

If  $X, Y$  are Banach spaces,  $T: X \rightarrow Y$  is a bounded linear operator [  $\|T\| = \sup_{\|x\| \leq 1, x \in X} \|Tx\| < \infty$  ]

If  $T$  is surjective, then  $T$  is open, i.e.

$T(B_r(x)) \supset B_r(y)$  for some  $r > 0$ , where  $B_r(x) = \{x \in X, \|x\| < r\}$

PMath 354 Measure Theory and Fourier Analysis Mar 16, 2009

Erratum: A5, Q4 (a)

$$\liminf_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(s) f(x-s) ds \geq M$$

Less Important                      Important

Note on notation: Usually  $\int_a^b f(t) dt$  denotes Lebesgue integral, though if  $f$  is Riemann integrable (i.e. piecewise cts & bdd) we may take this to be Riemann integral as well.

Major exception to this "rule": Vector-valued integrals

i.e.  $\int_{-\pi}^{\pi} h(s) s^* x ds$  will always be Riemann integrals.

Open Mapping Theorem:  $X, Y$  Banach spaces and  $T: X \rightarrow Y$  is

a bounded linear operator, which is surjective then  $T$  is open.

$G \subset X$  is an open set then  $T(G) \subset Y$  is open.

Notes:  $B_1(x) = \{x \in X : \|x - 0\| < 1\}$  - open

$T$  linear, bdd surjective

$T(B_1(x))$  open and contain  $0 = T(0)$

open Map Theorem

Hence, there is  $\delta > 0$  st.

$$T(B_1(x)) \supset B_\delta(y) = \{y \in Y : \|y - 0\| < \delta\}$$

$$r = \frac{1}{\delta}, \quad rT(B_1(x)) \supset rB_\delta(y)$$

$$\Rightarrow T(B_r(x)) \supset B_1(y)$$

corollary (to open mapping Thm)

$$A(\mathbb{Z}) = \{(c_n)_{n \in \mathbb{Z}} : f \in L_1(\mathbb{T})\} \stackrel{\text{def}}{=} C_0(\mathbb{Z}) \\ = \{(c_n)_{n \in \mathbb{Z}} : \lim_{|n| \rightarrow \infty} c_n = 0\}$$

proof  $(L_1(\mathbb{T}), \|\cdot\|_1)$  is a Banach space

$$(C_0(\mathbb{Z}), \|\cdot\|_\infty) \text{ where } \|(c_n)_{n \in \mathbb{Z}}\|_\infty = \sup_{n \in \mathbb{Z}} |c_n| =$$

$$\max_{n \in \mathbb{Z}} |c_n|$$

Define  $T: L_1(\mathbb{T}) \rightarrow C_0(\mathbb{Z})$  by  $T(f) = (c_n(f))_{n \in \mathbb{Z}} \in C_0(\mathbb{Z})$  by R-LT Lemma

Clearly  $T$  is linear, and  $f \in L_1(\mathbb{T})$  then

$$\|T(f)\|_\infty = \|(c_n(f))_{n \in \mathbb{Z}}\|_\infty = \max_{n \in \mathbb{Z}} |c_n(f)| \leq \|f\|_1$$

$$\text{Thus } \|T\| = \sup \{ \|T(f)\|_\infty : f \in L_1(\mathbb{T}), \|f\|_1 \leq 1 \} \leq 1$$

lemma prior to Riemann Lebesgue

If  $A(\mathbb{Z}) \stackrel{\text{def}}{=} T(L_1(\mathbb{T}))$  is equal to  $C_0(\mathbb{Z})$  so  $T$  is bounded

then the open mapping Thm tells us there is  $\delta > 0$ , st.

$$T(B_\delta(L_1(\mathbb{T}))) \supset B_{\frac{\delta}{2}}(C_0(\mathbb{Z}))$$

$$e^k = e^{ikt}$$

Let  $D_n = \sum_{j=-n}^n e^{ijt}$  be the  $n$ th Dirichlet kernel.

We saw that  $L_n = \|D_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n| \rightarrow \infty$  as  $n \rightarrow \infty$

However,  $T(D_n) = (C_k(D_n))_{k \in \mathbb{Z}} = (\dots, 0, \dots, 0, \underbrace{1, 1}_{\text{position } n}, \dots, 1, 0, \dots)$

$$C_k(D_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n e^{-ik} = \frac{1}{2\pi} \sum_{j=-n}^n \int_{-\pi}^{\pi} e^{i(j-k)t} dt$$

$$\|T(D_n)\|_{\infty} = \|(\dots, 0, \dots, 0, \dots, 1, 0, \dots)\|_{\infty} = 1$$

all zero unless  $j=k$  then 2i

and  $T(D_n) \in B_{1+\epsilon}(C_0(\mathbb{Z}))$  However,

$\{D_n\}_{n=1}^{\infty} \not\subset B_r(L_1(\mathbb{T}))$  for any fixed  $r > 0$

This contradicts the open mapping Theorem thus  $T$  can not be surjective.  $\square$

On the pointwise convergence of naïve Fourier series

Recall  $f \in L(\mathbb{T})$  then

non-Cesaro sums

$t \in \mathbb{R}$  (usually  $t \in [-\pi, \pi]$ )

$$S_n(f, t) = \sum_{j=-n}^n C_j(f) e^{ijt} = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) f(t-s) ds$$

from A5, Lebesgue integrable  
Understand this as

$$\text{Also, } D_n(s) = \begin{cases} \frac{\sin(n+\frac{1}{2})s}{\sin\frac{1}{2}s} & s \neq 0 \\ n+1 & s=0 \end{cases} \text{ for } s \in [-\pi, \pi]$$

Lemma: If  $f \in L(\mathbb{T})$ , satisfies  $\int_{-\pi}^{\pi} \left| \frac{f(t)}{t} \right| dt < \infty$

then  $S_n(f, 0) \xrightarrow{n \rightarrow \infty} 0$

proof: We have

$$S_n(f, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})s}{\sin\frac{1}{2}s} f(0-s) ds$$

$= D_n(s)$  a.e.  $f(-s)$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})s}{\sin\frac{1}{2}s} f(s) ds \quad [ \text{Inversion Invariance} ]$$

$$D_n(s) = D_n(-s)$$

Recall  $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$$\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} = \frac{\sin(nt) \cdot \cos\left(\frac{1}{2}t\right) + \cancel{\sin\left(\frac{1}{2}t\right) \cos(nt)}}{\sin\left(\frac{1}{2}t\right) \cdot \cancel{\sin\left(\frac{1}{2}t\right)}}$$

Thus

$$S_n(f; 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} f(t) dt$$

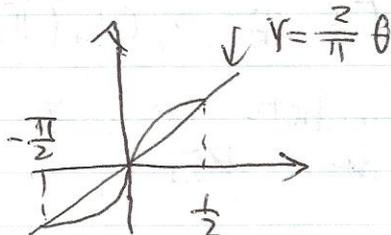
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos\left(\frac{1}{2}t\right)}{\sin\left(\frac{1}{2}t\right)} \cdot f(t) \sin(nt) dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(nt) \cdot f(t) dt$$

$n \rightarrow \infty$ , goes to 0 R-L Lemma

Let us estimate

$$\int_{-\pi}^{\pi} \left| \frac{\cos\left(\frac{1}{2}t\right)}{\sin\left(\frac{1}{2}t\right)} f(t) \right| dt \leq \int_{-\pi}^{\pi} \left| \frac{f(t)}{\sin\frac{1}{2}t} \right| dt \leq \int_{-\pi}^{\pi} \pi \left| \frac{f(t)}{t} \right| dt < \infty$$

$|\cos(\frac{1}{2}t)| \leq 1$



Recall  $|\sin\theta| \geq \frac{2}{\pi} |\theta|$ , for  $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \left| \sin\frac{1}{2}t \right| \geq \frac{2}{\pi} \left| \frac{1}{2}t \right| = \frac{1}{\pi} |t| \quad -\pi \leq t \leq \pi$$

$$\Rightarrow \frac{1}{\left| \sin\frac{1}{2}t \right|} \leq \pi \cdot \frac{1}{|t|} \quad \text{constant } C$$

Thus  $t \mapsto \frac{\cos\frac{1}{2}t}{\sin\frac{1}{2}t} f(t)$  is in  $L^1(\pi)$

so by Riemann-Lebesgue Lemma, we have

(corollary)

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} \frac{\cos\frac{1}{2}t}{\sin\frac{1}{2}t} f(t) \sin(nt) dt = 0$$

$$S_n(f; 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos\frac{1}{2}t}{\sin\frac{1}{2}t} f(t) \sin(nt) dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt$$

$\xrightarrow{n \rightarrow \infty} 0 + 0 = 0 \quad \square$

Mar 18, 2009 AMath 354

Last time: we saw

Lemma: If  $f \in L^1(\pi)$ ,  $\int_{-\pi}^{\pi} \left| \frac{f(t)}{t} \right| dt < \infty$

then  $\lim_{n \rightarrow \infty} S_n(f; 0) = 0$

Localization principle: Let  $f \in L(\mathbb{T})$ , suppose there is an open interval  $I \subset \mathbb{R}$  st.

$f(x) = 0$  for ae.  $x \in I$ . Then

for  $x \in I$ ,

$$\lim_{n \rightarrow \infty} S_n(f, x) = 0$$

proof: If fix  $x \in I$ , let  $g = x * \check{f}$ , so

$$g(s) = \check{f}(s-x) = f(x-s) \text{ for } s \in \mathbb{R}$$

~~Thus, there notice that I~~

There is a small open interval  $(-\delta, \delta)$  ( $\delta > 0$ ) about 0 st  $s \in (-\delta, \delta) \Rightarrow x-s \in I$ , which means that

$$g(s) = f(x-s) = 0 \text{ for ae. } s \in (-\delta, \delta) \quad g(t) = 0 \text{ a.e. } t \in [-\delta, \delta]$$

$$\text{Then } \int_{-\pi}^{\pi} \left| \frac{g(t)}{t} \right| dt = \int_{-\pi}^{-\delta} \left| \frac{g(t)}{t} \right| dt + \int_{-\delta}^{\delta} \left| \frac{g(t)}{t} \right| dt + \int_{\delta}^{\pi} \left| \frac{g(t)}{t} \right| dt$$

$$\stackrel{|t| \geq \delta}{\leq} \frac{1}{\delta} \int_{-\pi}^{-\delta} |g(t)| dt + 0 + \frac{1}{\delta} \int_{\delta}^{\pi} |g(t)| dt$$

$$\stackrel{\frac{1}{|t|} \leq \frac{1}{\delta}}{\leq} \frac{2}{\delta} \int_{-\pi}^{\pi} |g(t)| dt < \infty \text{ by assumption that } g \in L(\mathbb{T})$$

By the previous Lemma,  $\lim_{n \rightarrow \infty} S_n(g, 0) = 0$

Now we have

$$S_n(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) f(x-s) ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) \check{f}(s-x) ds$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) x * \check{f}(s) ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) \cdot g(s) ds$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) g(-s) ds \quad (\text{Z-inversion invariance})$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overset{\substack{\uparrow \\ \text{even } D_n(s) = D_n(-s)}}{D_n(s)} g_n(0-s) ds = S_n(g, 0) \xrightarrow[n \rightarrow \infty]{\text{cas above}} 0$$

Hence  $S_n(f, x) \xrightarrow{n \rightarrow \infty} 0$  □

Corollary: If  $f, g \in L(\mathbb{T})$  and  $I \subset \mathbb{R}$  is an open interval for which  $f(x) = g(x)$  for ae.  $x \in I$ .

then:  $\lim_{n \rightarrow \infty} S_n(f, x)$  exists  $\Leftrightarrow \lim_{n \rightarrow \infty} S_n(g, x)$  exists.

and the limits coincide. for  $x \in I$ .

proof: Let  $h = f - g$  apply localization principle  $\square$

Dini's Theorem: if  $f \in L(\mathbb{T})$  and  $f$  is differentiable at  $x \in [-\pi, \pi]$

Then  $\lim_{n \rightarrow \infty} S_n(f, x) = f(x)$ .

proof: Let  $f'(x)$  denote the derivative of  $f$  at  $x$ .

Given  $\varepsilon > 0$ , there is  $\delta > 0$  st.

$$|s-x| < \delta \Rightarrow \left| \frac{f(x-s) - f(x)}{-s} - f'(x) \right| < \varepsilon$$

Thus if  $|s| < \delta$ , we have

$$\left| \frac{f(x-s) - f(x)}{-s} - f'(x) \right| < \varepsilon \Rightarrow \left| \frac{f(x-s) - f(x)}{-s} \right| < \varepsilon + |f'(x)|$$

so  $s \mapsto \frac{f(x-s) - f(x)}{-s}$  is bounded (by  $\varepsilon + |f'(x)|$ ) on  $(-\delta, \delta)$

Let  $g(s) = x * \overset{\vee}{f}(s) - f(x)$  i.e.  $\underbrace{\quad}_{=c}$

$$g(s) = x * \overset{\vee}{f} - f(x) \quad \text{i.e.} \quad g(s) = x * \overset{\vee}{f} - \underbrace{f(x)}_{\text{const}}$$

Then  $g \in L(\mathbb{T})$ . Moreover.

$$\int_{-\pi}^{\pi} \left| \frac{g(t)}{t} \right| dt = \int_{-\pi}^{-\delta} \left| \frac{g(t)}{t} \right| dt + \int_{-\delta}^{\delta} \left| \frac{g(t)}{t} \right| dt + \int_{\delta}^{\pi} \left| \frac{g(t)}{t} \right| dt$$

$$\leq \frac{2}{\delta} \int_{-\pi}^{\pi} |g(t)| dt + \int_{-\delta}^{\delta} c \cdot dt < \infty$$

Like in proof of

$$= 2\delta c$$

Localization

Thus by the lemma, we have  $\lim_{n \rightarrow \infty} S_n(g, 0) = 0$

However, as in the proof above

$$S_n(g, 0) = S_n(x * \overset{\vee}{f} - f(x), 0) = S_n(x * \overset{\vee}{f}, 0) - S_n(f(x), 0)$$

$$= c_0(f(x)) + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dt = f(x)$$

$$= s_n(f, x) - f(x)$$

Thus  $\lim_{n \rightarrow \infty} (s_n(f, x) - f(x)) = 0$  □

### Inner product and Hilbert Spaces

Defn. Let  $X$  be a  $\mathbb{C}$ -valued Vector space,

An inner product  $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}$  is a map which satisfies

(i)  $\langle f, f \rangle \geq 0$  (non-negativity)  $f, g, h \in X$

(ii)  $\langle f, f \rangle = 0 \iff f = 0$  (non-degeneracy)  $\alpha \in \mathbb{C}$

(iii)  $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$  (additivity)

(iv)  $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$  (scalar homogeneity)

(v)  $\langle g, f \rangle = \overline{\langle f, g \rangle}$  (complex, conj.) (skew-symmetry)

Note (iii) & (v)  $\implies$

$$\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle$$

$$(iv) \& (v) \implies \langle f, \alpha g \rangle = \overline{\alpha} \langle f, g \rangle$$

We call the pair  $(X, \langle \cdot, \cdot \rangle)$  an inner product space.

If  $f \in X$ , we define

$$\|f\| = \sqrt{\langle f, f \rangle}$$

### Cauchy-Schwarz Inequality

If  $X$  is an inner product space and  $f, g \in X$ , then

$$|\langle f, g \rangle| \leq \|f\| \|g\|$$

Proof: If  $t \in \mathbb{R}$  then  $\bar{t} = t$  we have that

$$\begin{aligned}
 0 &\leq \langle tf + \bar{t}g, tf + \bar{t}g \rangle \stackrel{\text{additivity (on each side)}}{=} \langle tf, tf \rangle + \langle t\bar{t}f, g \rangle + \langle g, t\bar{t}f \rangle + \langle \bar{t}g, \bar{t}g \rangle \\
 &= t^2 \|f\|^2 + t \langle f, g \rangle + t \overline{\langle f, g \rangle} + \|g\|^2
 \end{aligned}$$

$$|x| \leq \sqrt{x^2} \leq \sqrt{x^2 + y^2} = |x+y|$$

$$= t^2 \|f\|^2 + 2t \operatorname{Re} \langle f, g \rangle + \|g\|^2$$

$$\leq t^2 \|f\|^2 + 2t |\langle f, g \rangle| + \|g\|^2 =: P(t)$$

we have

$P(t)$  is a quadratic function from  $\mathbb{R}$  to  $\mathbb{R}$  with  $P(t) \geq 0$

Thus the discriminant

$$(2|\langle f, g \rangle|)^2 - 4\|f\|^2\|g\|^2 \leq 0$$

$$\Rightarrow |\langle f, g \rangle| \leq \|f\| \|g\| \quad \square$$

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Last time

Cauchy-Schwarz Inequality

Note  $\|f\| = \sqrt{\langle f, f \rangle}$

If  $(X, \langle \cdot, \cdot \rangle)$  is an inner product space,  $f, g \in X$ , then

$$|\langle f, g \rangle| \leq \|f\| \|g\|$$

proof: if  $t \in \mathbb{R}$

$$0 \leq \langle tf+g, tf+g \rangle \quad [\text{work here}]$$

$$\leq t^2 \|f\|^2 + 2t |\langle f, g \rangle| + \|g\|^2 =: P(t)$$

$$t^2 a + t b + c$$

quadratic  $P: \mathbb{R} \rightarrow \mathbb{R}$

$$P(t) \geq 0 \Rightarrow \text{discriminant } (2|\langle f, g \rangle|)^2 - 4\|f\|^2\|g\|^2 \leq 0$$

Note: We have that  $|\langle f, g \rangle| = \|f\| \|g\| \Leftrightarrow f = tg$  for some  $t \geq 0$

if  $|\langle f, g \rangle| = \|f\| \|g\|$ , then the discriminant of  $P(t)$  above must be 0.

so for  $t_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2|\langle f, g \rangle|}{2\|f\|^2}$

we have  $0 \leq \langle t_0 f + g, t_0 f + g \rangle \leq 0$  so  $\langle t_0 f + g, t_0 f + g \rangle = 0$

which implies  $t_0 f + g = 0$

Thus  $-t_0 f = g$

$$\Rightarrow \frac{|\langle f, g \rangle|}{\|f\|^2} f = g$$

$$\Rightarrow \frac{\|f\| \|g\|}{\|f\|^2} f = g \Rightarrow g = \frac{\|f\|}{\|g\|} f$$

$\geq 0$   
we may assume  $\|f\| \|g\| \neq 0$  for non-triviality

z.f.  $f = tg$  for  $t \geq 0$  then

$$|\langle f, tg \rangle| = t |\langle f, g \rangle|$$

$$|\langle f, g \rangle| = |\langle tg, g \rangle| = t |\langle g, g \rangle| = t \|g\|^2 = \|tg\| \|g\| \\ = \|f\| \|g\|$$

Corollary (to Cauchy - Schwarz Inequality)

z.f.  $(X, \langle \cdot, \cdot \rangle)$  is an inner product space, then  $\|f\| = \sqrt{\langle f, f \rangle}$  defines a norm on  $X$

proof: z.f.  $f \in X$ ,  $\langle f, f \rangle \geq 0 \Rightarrow \sqrt{\langle f, f \rangle} \geq 0$  and

$$\langle f, f \rangle = 0 \Leftrightarrow f = 0$$

$$\text{So } \|f\| = 0 \Leftrightarrow f = 0$$

$$\text{z.f. } f \in X, \alpha \in \mathbb{C} \text{ then } \|\alpha f\| = \sqrt{\langle \alpha f, \alpha f \rangle} = \sqrt{\alpha \bar{\alpha} \langle f, f \rangle} = \sqrt{|\alpha|^2 \langle f, f \rangle} \\ = |\alpha| \sqrt{\langle f, f \rangle} = |\alpha| \|f\|$$

z.f.  $f, g \in X$ , we have

$$\|f+g\|^2 = \langle f+g, f+g \rangle = \|f\|^2 + 2 \operatorname{Re} \langle f, g \rangle + \|g\|^2 \quad \left( \begin{array}{l} \text{like in} \\ \text{proof of} \\ \text{Cauchy-Schwarz} \end{array} \right)$$

$$\leq \|f\|^2 + 2 |\langle f, g \rangle| + \|g\|^2$$

$$\leq \|f\|^2 + 2 \|f\| \|g\| + \|g\|^2 = (\|f\| + \|g\|)^2$$

$$\Rightarrow \|f+g\| \leq \|f\| + \|g\| \quad \square$$

Define A Hilbert space is an inner product space

$(H, \langle \cdot, \cdot \rangle)$  st.  $H$  is complete wrt.  $\|\cdot\|$ .

Examples: (i)  $\mathbb{C}^n$ ,  $x = (x_1, \dots, x_n)$   $y = (y_1, \dots, y_n)$

$$\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i \rightarrow \text{inner product}$$

since  $\mathbb{C}^n \cong \mathbb{R}^{2n}$   $\mathbb{C}^n$  is complete and thus a topologically

Hilbert space.

(ii) Let  $A \in \mathcal{L}(\mathbb{R})$ , [Lebesgue measurable set]

$$\lambda(A) > 0$$

On  $L_2(A)$  define

$$\langle f, g \rangle = \int_A f \bar{g}$$

Note by Hölder's inequality since  $\bar{g} \in L_2(A)$  if  $g \in L_2(A)$ .

we find  $f\bar{g} \in L_1(A)$

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_A f \bar{f}} = \left[ \int_A |f|^2 \right]^{1/2} = \|f\|_2$$

We saw that  $L_2(A)$  is  $\|\cdot\|_2$ -complete. Thus

$(L_2(A), \langle \cdot, \cdot \rangle)$  is a Hilbert space.

(ii')  $L_2(\mathbb{T}) = \{ f: \mathbb{R} \rightarrow \mathbb{C} \mid f \in M_{\mathbb{C}}(\mathbb{R}) \text{ a.e. } 2\pi\text{-periodic} \}$   
 $\int_{\mathbb{T}} |f|^2 < \infty$

$$\text{Inner product } \langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f \bar{g}$$

(iii)  $C(\mathbb{T}) = \{ f: \mathbb{R} \rightarrow \mathbb{C} \mid f \text{ is cts } 2\pi\text{-periodic} \}$

$$\text{Inner product } \langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f \bar{g} \quad (\text{Riemann integral})$$

Note  $(C(\mathbb{T}), \|\cdot\|_2)$  is NOT complete

since by A variant A4. Q2, this space is dense in  $L_2(\mathbb{T})$ .

Thus  $(C(\mathbb{T}), \langle \cdot, \cdot \rangle)$  is an inner product space though NOT a Hilbert space

(iv)  $L_2(\mathbb{Z}) = \{ (c_n)_{n \in \mathbb{Z}} : c_n \in \mathbb{C}, \text{ and } \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty \}$

$\mathbb{Z}$ -indexed "square summable sequences"

Inner product  $x = (x_n)_{n \in \mathbb{Z}}, y = (y_n)_{n \in \mathbb{Z}} \in L_2(\mathbb{Z})$

$$\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x_n \bar{y}_n$$

We should verify that this sum converges. We will establish

absolute convergence.

$$\sum_{n=-\infty}^{\infty} |x_n \bar{y}_n| = \lim_{N \rightarrow \infty} \sum_{n=-N}^N |x_n \bar{y}_n| \leq \lim_{N \rightarrow \infty} \left( \sum_{n=-N}^N |x_n|^2 \right)^{1/2} \left( \sum_{n=-N}^N |y_n|^2 \right)^{1/2}$$

$$\begin{aligned} & \in L_2(\mathbb{Z}) \quad \in L_2(\mathbb{Z}) \\ & = \left( \sum_{n=-\infty}^{\infty} |x_n|^2 \right)^{1/2} \left( \sum_{n=-\infty}^{\infty} |y_n|^2 \right)^{1/2} < \infty \end{aligned}$$

Since  $\left( \sum_{n=-N}^N |x_n \bar{y}_n| \right)_{N=1}^{\infty}$  is increasing and bounded above, the

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N |x_n \bar{y}_n| \text{ exists.}$$

Now we MUST establish  $l_2(\mathbb{Z})$  is a vector space

$$(x_n)_{n \in \mathbb{Z}} + (y_n)_{n \in \mathbb{Z}} = (x_n + y_n)_{n \in \mathbb{Z}}$$

$$\alpha(x_n)_{n \in \mathbb{Z}} = (\alpha x_n)_{n \in \mathbb{Z}}$$

$$\sum_{n=-\infty}^{\infty} |x_n + y_n|^2 = \lim_{N \rightarrow \infty} \sum_{n=-N}^N |x_n + y_n|^2$$

$$\leq \lim_{N \rightarrow \infty} \sum_{n=-N}^N (|x_n|^2 + 2|x_n y_n| + |y_n|^2)$$

$$= \lim_{N \rightarrow \infty} \left( \sum_{n=-N}^N |x_n|^2 + 2 \sum_{n=-N}^N |x_n y_n| + \sum_{n=-N}^N |y_n|^2 \right)$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} |x_n|^2}_{< \infty} + \underbrace{\sum_{n=-\infty}^{\infty} |x_n \bar{y}_n|}_{< \infty} + \underbrace{\sum_{n=-\infty}^{\infty} |y_n|^2}_{< \infty} = |x_n| |y_n| = |x_n \bar{y}_n| = |x_n y_n|$$

$$< \infty$$

clearly,  $(\alpha x_n)_{n \in \mathbb{Z}} \in L_2(\mathbb{Z})$  if  $\alpha \in \mathbb{C}$

$$(x_n)_{n \in \mathbb{Z}} \in l_2(\mathbb{Z}).$$

$(l_2(\mathbb{Z}), \langle \cdot, \cdot \rangle)$  is an inner product space.

Fact:  $(l_2(\mathbb{Z}), \|\cdot\|_2)$  is complete and hence a Hilbert space.

$$\|(x_n)_{n \in \mathbb{Z}}\|_2 = \left( \sum_{n=-\infty}^{\infty} |x_n|^2 \right)^{1/2} \quad \text{very hard/tricky to prove}$$

see notes for PMath 351.

The proof follows Plancherel Theorem.

Hilroy

Mar 23, 2009 PMath 354 Measure Theory and Fourier Analysis

Definition: Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner-product space.

A family  $\{f_i\}_{i \in I} \in X$  is called orthogonal if  $\langle f_i, f_j \rangle = 0$

for  $i \neq j$ . Moreover,  $\{f_i\}_{i \in I}$  is called orthogonal if  $\langle f_i, f_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$  (orthogonal) Ortho-normal (normal)

prop. If  $\{f_1, \dots, f_n\} \subset X$  is orthogonal, then

$$\|f_1 + \dots + f_n\|^2 = \|f_1\|^2 + \dots + \|f_n\|^2 \quad (\text{Pythagorean formula})$$

proof:  $\| \sum_{k=1}^n f_k \|^2 = \langle \sum_{k=1}^n f_k, \sum_{l=1}^n f_l \rangle = \sum_{k=1}^n \sum_{l=1}^n \langle f_k, f_l \rangle = \sum_{k=1}^n \langle f_k, f_k \rangle$   
 $= \sum_{k=1}^n \|f_k\|^2 \quad \square = 0 \text{ unless } k=l$

Linear Approximation Lemma:

Let  $\{e_1, \dots, e_n\}$  be an orthonormal set in an inner product space

$(X, \langle \cdot, \cdot \rangle)$  If  $f \in X$  then  $\text{dist}(f, E_n) := \inf \{ \|f - \sum_{i=1}^n \alpha_i e_i\|, \text{st. } \alpha_1, \dots, \alpha_n \in \mathbb{C} \}$

Let  $E_n = \text{span}\{e_1, \dots, e_n\}$

$$= \|f - \sum_{i=1}^n \langle f, e_i \rangle e_i\| = \left( \|f\|^2 - \sum_{i=1}^n |\langle f, e_i \rangle|^2 \right)^{1/2}$$

Proof: Arbitrary elements of  $E_n$  are of the form  $g = \sum_{i=1}^n \alpha_i e_i$

Let us estimate

$$\|f - g\|^2 = \langle f - g, f - g \rangle = \|f\|^2 - 2 \text{Re} \langle f, g \rangle + \|g\|^2$$

$$= \|f\|^2 - 2 \text{Re} \left[ \langle f, \sum_{i=1}^n \alpha_i e_i \rangle \right] + \left\| \sum_{i=1}^n \alpha_i e_i \right\|^2$$

Pythagoras  $\|e_i\|^2 = 1$

$$= \|f\|^2 - 2 \text{Re} \left[ \sum_{i=1}^n \bar{\alpha}_i \langle f, e_i \rangle \right] + \sum_{i=1}^n |\alpha_i|^2$$

(f)  $\geq \|f\|^2 - 2 \left| \sum_{i=1}^n \bar{\alpha}_i \langle f, e_i \rangle \right| + \sum_{i=1}^n |\alpha_i|^2$

(ff)  $\geq \|f\|^2 - 2 \left( \sum_{i=1}^n |\bar{\alpha}_i|^2 \right)^{1/2} \left( \sum_{i=1}^n |\langle f, e_i \rangle|^2 \right)^{1/2} + \sum_{i=1}^n |\alpha_i|^2$

(Cauchy Schwarz)

(Complete the square)

$$\begin{aligned}
&= \|f\|^2 - \sum_{i=1}^n |\langle f, e_i \rangle|^2 + \sum_{i=1}^n |\langle f, e_i \rangle|^2 - 2 \left( \sum_{i=1}^n |\alpha_i|^2 \right)^{1/2} \left( \sum_{i=1}^n |\langle f, e_i \rangle|^2 \right)^{1/2} \\
&\quad + \sum_{i=1}^n |\alpha_i|^2 \\
&= \|f\|^2 - \sum_{i=1}^n |\langle f, e_i \rangle|^2 + \left( \left( \sum_{i=1}^n |\langle f, e_i \rangle|^2 \right)^{1/2} - \left( \sum_{i=1}^n |\alpha_i|^2 \right)^{1/2} \right)^2
\end{aligned}$$

$$(\text{f}) \geq \|f\|^2 - \sum_{i=1}^n |\langle f, e_i \rangle|^2$$

Conclusion:

$$\begin{aligned}
\text{dist}(f, E_n)^2 &= \inf \left\{ \|f - \sum_{i=1}^n \alpha_i e_i\|^2, \alpha_1, \dots, \alpha_n \in \mathbb{C} \right\} \\
&\quad (\text{from above}) \geq \|f\|^2 - \sum_{i=1}^n |\langle f, e_i \rangle|^2
\end{aligned}$$

This inequality is equality (ies)

$$(\text{f}) \quad \sum_{i=1}^n \bar{\alpha}_i \langle f, e_i \rangle \in \mathbb{R}$$

(ff) exactly when  $\alpha_i = \langle f, e_i \rangle \quad i=1, \dots, n \quad \begin{matrix} \Rightarrow (\text{f}) \\ \Rightarrow (\text{fff}) \end{matrix}$

$$(\text{fff}) \quad \sum_{i=1}^n |\alpha_i|^2 = \sum_{i=1}^n |\langle f, e_i \rangle|^2 \text{ which follows from (ff)}$$

Thus we find that all  $\geq$ 's are = 's exactly when  $\alpha_i = \langle f, e_i \rangle \quad i=1, \dots, n$

$$\begin{aligned}
\text{dist}(f, E_n)^2 &= \|f - \sum_{i=1}^n \langle f, e_i \rangle e_i\|^2 \\
&= \|f\|^2 - \sum_{i=1}^n |\langle f, e_i \rangle|^2 \quad \square
\end{aligned}$$

### Ortho-normal Basis Theorem

Let  $X$  be an infinite-dimensional inner product space, and  $\{e_i\}_{i=1}^{\infty}$  is an ortho-normal set in  $X$ .

Then TFAE

- (i)  $\text{span} \{e_i\}_{i=1}^{\infty} := \left\{ \sum_{i=1}^n \alpha_i e_i \mid n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \mathbb{C} \right\}$
- $:= \bigcup_{n=1}^{\infty} \text{span} \{e_1, \dots, e_n\}$  is dense in  $X$
- [dense wrt.  $\|f\| = \langle f, f \rangle^{1/2}$ ]

(ii) for every  $f \in X$   
 $\|f\|^2 = \sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2$  (Bessel's equality)

(iii) For every  $f \in X$   
 $f = \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i$  i.e.  $\lim_{n \rightarrow \infty} \|f - \sum_{i=1}^n \langle f, e_i \rangle e_i\| = 0$

(iv) For every  $f, g \in X$   
 $\langle f, g \rangle = \sum_{i=1}^{\infty} \langle f, e_i \rangle \langle e_i, g \rangle$  (Parseval's identity)

Note By (iii) we often call the numbers  $\langle f, e_i \rangle$  as (abstract) Fourier coefficients.

proof: (i)  $\Leftrightarrow$  (iii) Let  $E_n = \text{span}\{e_1, \dots, e_n\}$

Notice that  $E_n \subset E_{n+1}$  for each  $n$ , and (so it follows)

Thus if  $(f \in X)$  (i)

$\text{Span}\{e_i\}_{i=1}^{\infty} = \bigcup_{n=1}^{\infty} E_n$  is dense in  $X \Leftrightarrow$

for every  $f \in X$ ,  $\text{dist}(f, \bigcup_{n=1}^{\infty} E_n) = 0$

$\Leftrightarrow$  for every  $f \in X$ ,  $\lim_{n \rightarrow \infty} \text{dist}(f, E_n) = 0$

$\Leftrightarrow \lim_{n \rightarrow \infty} \left( \|f\|^2 - \sum_{i=1}^n |\langle f, e_i \rangle|^2 \right) = 0$  by Linear Approximation Lemma  
 (iii)

(ii)  $\Leftrightarrow$  (iii)

By Linear Approximation Lemma

$\|f - \sum_{i=1}^n \langle f, e_i \rangle e_i\|^2 = \|f\|^2 - \sum_{i=1}^n |\langle f, e_i \rangle|^2$  for all

$f \in X$ , Thus

$\lim_{n \rightarrow \infty} \|f - \sum_{i=1}^n \langle f, e_i \rangle e_i\|^2 = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \sum_{i=1}^n |\langle f, e_i \rangle|^2 = \|f\|^2$

(ii)

(iii)

⇒ (2v)

(iii) Let us fix  $g \in X$ , Note that  $T_g : X \rightarrow \mathbb{C}$

$T_g(f) = \langle f, g \rangle$  is a bounded linear function. Indeed, it is linear. Moreover

$$\begin{aligned} \|T_g\|_X &= \sup \{ |T_g(f)| : f \in X, \|f\| \leq 1 \} \\ &= \sup \{ |\langle f, g \rangle|, f \in X, \|f\| \leq 1 \} \leq \|g\| \\ &\leq \|f\| \|g\| \leq \|g\| \end{aligned}$$

In particular,  $T_g : X \rightarrow \mathbb{C}$  is cts. continuity

Thus  $T_g(f) = T_g \left( \lim_{n \rightarrow \infty} \underbrace{\sum_{i=1}^n \langle f, e_i \rangle e_i}_{\text{by (iii)}} \right) \stackrel{\text{continuity}}{=} \lim_{n \rightarrow \infty} \sum_{i=1}^n \langle f, e_i \rangle T_g(e_i)$

hence (2v)  $= \lim_{n \rightarrow \infty} \sum_{i=1}^n \langle f, e_i \rangle \langle e_i, g \rangle$

(iv) ⇒ (iii) we have for  $f \in X$

$$\|f\|^2 \langle f, f \rangle = \sum_{i=1}^{\infty} \langle f, e_i \rangle \langle e_i, f \rangle = \sum_{i=1}^{\infty} \langle f, e_i \rangle \overline{\langle f, e_i \rangle} = \sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2$$

Mar 25, 2009 PMath 354 Fourier Analysis VIII

Last time  $X$  inner product space  
 $\{e_n\}_{n=1}^{\infty}$  an orthonormal sequence in  $X$  TFAE

- (i)  $\text{span} \{e_n\}_{n=1}^{\infty} = \bigcup_{n=1}^{\infty} \text{span} \{e_k\}_{k=1}^n$  dense in  $X$
- (ii)  $f \in X \Rightarrow \langle f, f \rangle = \|f\|^2 = \sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2$  (Bessel's identity)
- (iii)  $f \in X \Rightarrow f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k$
- (iv)  $f, g \in X \Rightarrow \langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle$  (Parseval's identity)

Note Bessel's Inequality

$\{e_k\}_{k=1}^{\infty}$  is an orthonormal sequence in an inner product space  $X$   
then for  $f \in X$

$$\langle f, f \rangle = \|f\|^2 \geq \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2$$

*Hilroy*

proof: if  $E_n = \text{span}\{e_1, \dots, e_n\}$  then

$$0 \leq \text{dist}(f, E_n)^2 \leq \|f\|^2 - \sum_{k=1}^n |\langle f, e_k \rangle|^2$$

Linear Approx. Lemma

Hence  $\sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2 = \sup_{n \in \mathbb{N}} \sum_{k=1}^n |\langle f, e_k \rangle|^2 \leq \|f\|^2 \quad \square$

Example:  $l_2(\mathbb{Z}) = \left\{ x = (x_n)_{n \in \mathbb{Z}} : \sum_{n \in \mathbb{Z}} |x_n|^2 < \infty \right\}$   
 $= \lim_{N \rightarrow \infty} \sum_{n=-N}^N |x_n|^2 < \infty \quad x_n \in \mathbb{C}, n \in \mathbb{Z}$

$\langle x, y \rangle = \sum_{n=-\infty}^{+\infty} x_n \overline{y_n}$  (sum always converges absolutely)

For  $n \in \mathbb{Z}$ , let  $e_n = \{ \dots, 0, 0, \underset{\substack{\uparrow \\ n\text{th position}}}{1}, 0, 0, \dots \}$

Now: if  $n, m \in \mathbb{Z}$

$$\langle e_n, e_m \rangle = \begin{cases} 1 & n=m \\ 0 & \text{otherwise} \end{cases}$$

so  $\{e_n\}_{n \in \mathbb{Z}}$  is an orthonormal set. Now if

$x = (x_n)_{n \in \mathbb{Z}}$ , we have  $\langle x, e_n \rangle = \sum_{k=-\infty}^{+\infty} x_k \overline{e_{n,k}} = x_n$   
 $= x_n \overline{1} = x_n$   $= 0$  unless  $k=n$

We have

$$\|x - \sum_{k=-n}^n \langle x, e_k \rangle e_k\|^2 = \|x - (0, 0, \dots, x_n, x_{-n+1}, \dots, x_n, 0, 0, \dots)\|^2$$

$$= \|(0, \dots, x_{-n-2}, x_{-n-1}, 0, 0, \dots, 0, 0, x_{n+1}, x_{n+2}, \dots)\|^2$$

$$= \sum_{k=-\infty}^{-n-1} |x_k|^2 + \sum_{k=n+1}^{\infty} |x_k|^2 \quad \leftarrow \text{tails of summable series.}$$

$$= \sum_{k=-\infty}^{\infty} |x_k|^2 - \sum_{k=-n}^n |x_k|^2 \xrightarrow{n \rightarrow \infty} \sum_{k=-\infty}^{\infty} |x_k|^2 - \sum_{k=-\infty}^{\infty} |x_k|^2 = 0$$

Thus  $x = \sum_{k=-\infty}^{\infty} \langle x, e_k \rangle e_k$  so by orthonormal basis theorem  $\{e_k\}_{k \in \mathbb{Z}}$  is dense in  $l_2(\mathbb{Z})$

### Hilbertian Fourier Analysis

$$L_2(\mathbb{T}) = \left\{ f: \mathbb{R} \rightarrow \mathbb{C} \mid \begin{array}{l} f \in M_{\mathbb{C}}(\mathbb{R}) \text{ a.e. } 2\pi\text{-periodic} \\ \int_{-\pi}^{\pi} |f|^2 < \infty \end{array} \right\} / \sim \text{a.e.}$$

norm  $\|f\| = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \right)^{1/2}$

inner product  $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f \bar{g}$

Theorem:  $\{e_k\}_{k \in \mathbb{Z}}$  where  $e_k(t) = e^{ikt}$  is an orthonormal basis, (i.e. orthonormal sequence st.  $\text{span} \{e_k\}_{k \in \mathbb{Z}}$  is dense)

in  $l_2(\mathbb{T})$

proof: we see that  $\{e^k\}_{k \in \mathbb{Z}}$  is an orthonormal set

if  $k, l \in \mathbb{Z}$

$$\begin{aligned} \langle e^k, e^l \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikt} \overline{e^{ilt}} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{e^{i(k-l)t}}_{\cos((k-l)t) + i\sin((k-l)t)} dt \\ &= \begin{cases} 1 & k=l \\ 0 & k \neq l \end{cases} \end{aligned}$$

We can offer two proofs that  $\text{span} \{e^k\}_{k \in \mathbb{Z}}$  is dense in  $l_2(\mathbb{T})$ .

(Abstract Summability Kernel Theorem) we saw as a corollary to the aforementioned theorem that

$$\lim_{n \rightarrow \infty} \|f - \sigma_n(f)\|_2 = 0$$

Notice  $\sigma_n(f) = \frac{1}{n+1} \sum_{j=0}^n \sum_{k=-j}^j c_k(f) e^k \in \text{span} \{e_k\}_{k=-n}^n$

Hence if  $f \in L_2(\mathbb{T})$  since  $E_n \subset E_{n+1}$   
 $\text{dist}(f, \text{span}\{e^k\}_{k \in \mathbb{Z}}) = \lim_{n \rightarrow \infty} \text{dist}(f, \underbrace{\text{span}\{e^k\}_{k=-n}^n}_{E_n})$   
 $\leq \lim_{n \rightarrow \infty} \|f - \sigma_n(f)\|_2 = 0$  ✓ closure

Hence  $\text{dist}(f, \text{span}\{e^k\}_{k \in \mathbb{Z}}) = 0$  so  $f \in \overline{\text{span}\{e^k\}_{k \in \mathbb{Z}}}$   
 (Stone-Weierstrass Theorem)

Let  $\text{Trig}(\mathbb{T}) = \underbrace{\text{span}\{e^k\}_{k \in \mathbb{Z}}}_{\text{trigonometric polynomials}} = \left\{ \sum_{k=-n}^n \alpha_k e^{ik} : n \in \mathbb{Z}, \alpha_k \in \mathbb{C} \right\}$   
 "polynomials" = Laurent polynomials

Since we have pointwise multiplication  $e^k \cdot e^l = e^{k+l}$

We note that

—  $\text{Trig}(\mathbb{T}) \subset C(\mathbb{T})$  is an algebra of functions.

—  $\text{Trig}(\mathbb{T})$  separates points in  $[-\pi, \pi)$

ie. if  $t \neq s$  in  $[-\pi, \pi)$  then there is  $k \in \mathbb{Z}$ ,  
 $e^{ikt} \neq e^{iks}$

—  $\text{Trig}(\mathbb{T})$  contains constant function 1

—  $\text{Trig}(\mathbb{T})$  is conjugation closed,  $\sum_{k=-n}^n \alpha_k e^{ik} = \sum_{k=-n}^n \overline{\alpha_k} e^{-ik} \in \text{Trig}(\mathbb{T})$

Thus by Stone-Weierstrass Theorem

$$\overline{\text{Trig}(\mathbb{T})}^{\|\cdot\|_\infty} = C(\mathbb{T})$$

Thus if  $f \in L_2(\mathbb{T})$  and  $\varepsilon > 0$ , find  $h \in C(\mathbb{T})$  st. then  
 $\|h - f\|_2 < \varepsilon/2$ , and find  $p \in \text{Trig}(\mathbb{T})$  st.  $\|p - h\|_\infty < \varepsilon/2$

$$\|f - p\|_2 \leq \|f - h\|_2 + \|h - p\|_2 \leq \|f - h\|_2 + \|h - p\|_\infty$$

$$\underbrace{\hspace{10em}}_{\text{check}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \square$$

Corollary: If  $f \in L_2(\mathbb{T})$ , then

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$$\lim_{n \rightarrow \infty} \|s_n(f) - f\|_2 = 0$$

proof: If  $k \in \mathbb{Z}$   $G_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt$   
 $= \frac{1}{2\pi} \int_{-\pi}^{\pi} f \overline{e^k} = \langle f, e^k \rangle$

since  $\{e^k\}_{k \in \mathbb{Z}}$  is an orthonormal basis for  $L_2(\mathbb{T})$ .  
We have

$$\|f - s_n(f)\| = \left\| f - \sum_{k=-n}^n G_k(f) e^k \right\|_2$$

$$= \left\| f - \sum_{k=-n}^n \langle f, e^k \rangle e^k \right\|_2 \xrightarrow{n \rightarrow \infty} 0$$

by orthonormal Basis Theorem  $\square$

Apr 3, 2009 DMath 354 Measure Theory and Fourier Analysis

Theorem (Gibbs phenomenon) If  $f \in L(\mathbb{T})$   
and there is  $t_0 \in \mathbb{R}$  (usually  $t_0 \in [-\pi, \pi]$ ) and  $\delta > 0$  st.

$f$  is piecewise <sup>boundedly</sup> differentiable on  $(t_0 - \delta, t_0 + \delta)$

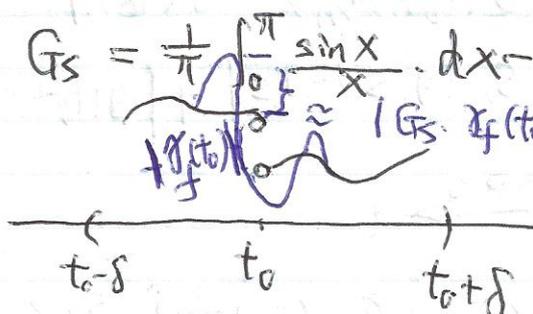
$f$  is continuous on  $(t_0 - \delta, t_0 + \delta)$  except at  $t_0$

Then  $\lim_{n \rightarrow \infty} [S_n(f, t_0 + \frac{\pi}{n}) - f(\frac{\pi}{n} + t_0)] = G_\delta \cdot \mathcal{J}_f(t_0)$

and  $\lim_{n \rightarrow \infty} [f(t_0 - \frac{\pi}{n}) - S_n(f, t_0 - \frac{\pi}{n})] = G_\delta \cdot \mathcal{J}_f(t_0)$

Recall  $G_\delta = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin x}{x} \cdot dx - \frac{1}{2} \approx 0.089$

Picture



proof: Let for  $t \in [-\pi, \pi]$

$$g(t) = \begin{cases} f(t) - \mathcal{J}_f(t_0) \cdot F(t - t_0) & t \neq t_0 \\ \mathcal{W}_f(t_0) & t = t_0 \end{cases}$$

As before,  $g$  is cts on  $(t_0 - \delta, t_0 + \delta)$  and

$g$  is piecewise boundedly differentiable on  $(t_0 - \delta, t_0 + \delta)$

Thus for  $t$  in any compact sub interval of  $(t_0 - \delta, t_0 + \delta)$   
(say  $t \in [t_0 - \delta/2, t_0 + \delta/2]$ ) we have

$$\lim_{n \rightarrow \infty} S_n(g, t) = g(t) \text{ uniformly for } t \in [t_0 - \delta/2, t_0 + \delta/2]$$

by the last corollary, we have

$$f = g + \mathcal{J}_f(t_0) \cdot F \quad \text{so}$$

(for  $t \in [t_0 - \delta/2, t_0 + \delta/2]$  we have)

$$\lim_{n \rightarrow \infty} S_n(f, t_0 + \frac{\pi}{n}) = \lim_{n \rightarrow \infty} [S_n(g, t_0 + \frac{\pi}{n}) + \mathcal{J}_f(t_0) \cdot \dots]$$

$S_n(F, (t_0 + \frac{\pi}{n}) - t_0)$  --- last class

Note that since

$\lim_{n \rightarrow \infty} S_n(g, t) = g(t)$  uniformly for  $t \in [t_0 - \delta/2, t_0 + \delta/2]$

we find for  $n, \frac{\pi}{n} \leq \delta/2$  we have

$$\begin{aligned} |S_n(g, t_0 + \frac{\pi}{n}) - S_n(g, t_0)| &\leq |S_n(g, t_0 + \frac{\pi}{n}) - g(t_0 + \frac{\pi}{n})| + \\ &\quad |g(t_0 + \frac{\pi}{n}) - g(t_0)| \\ &\leq \sup_{s \in [t_0 - \delta/2, t_0 + \delta/2]} |S_n(g, s) - g(s)| + |g(t_0 + \frac{\pi}{n}) - g(t_0)| \\ &\xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

$$\text{So } \lim_{n \rightarrow \infty} S_n(f, t_0 + \frac{\pi}{n}) = g(t_0) + \sigma_f(t_0) \underbrace{\frac{1}{\pi} \int_0^{\pi} \frac{\sin x}{x} dx}_{\text{from last lemma}}$$

$$= \omega_f(t_0) + \sigma_f(t_0) \left( G_s + \frac{1}{2} \right)$$

$$= \frac{1}{2} (f(t_0^+) + f(t_0^-)) + (f(t_0^+) - f(t_0^-)) \left( G_s + \frac{1}{2} \right)$$

$$= f(t_0^+) + \sigma_f(t_0) \cdot G_s$$

$$\text{Thus } \lim_{n \rightarrow \infty} [S_n(f, t_0 + \frac{\pi}{n}) - f(t_0 + \frac{\pi}{n})]$$

$$= \cancel{f(t_0^+)} + \sigma_f(t_0) G_s - \cancel{f(t_0^+)}$$

$$= \sigma_f(t_0) G_s$$

The limit from the left is similar  $\square$

Recall Fejers Thm. If  $f \in L(\pi)$  and  $t \in \mathbb{R}$  ( $t \in (-\pi, \pi)$ )

is st.  $\omega_f(t)$  exists, then

$$\lim_{n \rightarrow \infty} \sigma_n(f, t) = \omega_f(t)$$

Moreover, if  $(a, b)$  is an open interval on which  $f$  is continuous, then for any compact subinterval  $[a', b'] \subset (a, b)$

We have

$$\lim_{n \rightarrow \infty} \sigma_n(f, t) = f(t) \text{ uniformly for } t \in [a', b']$$

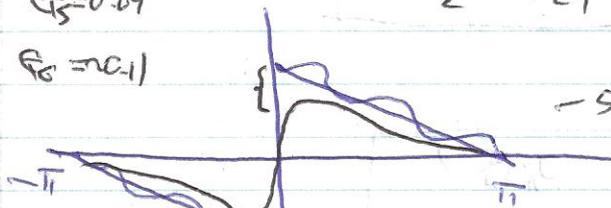
Reasons why we used "compact subinterval" is

$$G_0 = \frac{1}{\pi} \int_0^\pi \left( \frac{\sin x}{x} - \ln x \right) dx \approx -0.11$$

Picture. 
$$F(t) = \begin{cases} \frac{1}{2} - \frac{t}{2\pi} & t \in (0, \pi) \\ -\frac{1}{2} - \frac{t}{2\pi} & t \in (-\pi, 0) \end{cases}$$

$$G_5 = 0.09$$

$$G_0 = -0.11$$



- see website

Lebesgue integral  
MCT Fatou. LDCT.

Holder Minkovsk.

$L_p$ -space  
• Complete

$C[a, b]$  dense  $L_p[a, b]$  ( $1 \leq p < \infty$ )  
 $C(\mathbb{T})$  dense  $L_p(\mathbb{T})$

Homogeneous  
Ban-space over  
 $\mathbb{T}$

Fourier series,  $S_n(f)$

Convolutions  
Dirichlet kernel

From Homogeneous block

From [Caib] block

Norms of functions  
Operators  
Convolution operators  
on  $L_p(\mathbb{T})$  ( $L_1(\mathbb{T})$  &  $C(\mathbb{T})$ )

Lebesgue constants

Banach-Steinhaus Thm

Non-convergence of Fourier Series  
in  $C(\mathbb{T}), L_1(\mathbb{T})$

non-pointwise convergence  
in  $C(\mathbb{T})$

- Cesaro sums  $\sigma_n(f)$
- Fejer kernel
- summability kernels

Abstract summability kernel Thm

- Convergence of  $\sigma_n(f)$  on  $C(\mathbb{T}), L_p(\mathbb{T})$  ( $p < \infty$ )
- $\sigma_n(f) = \sigma_n(g) \iff f = g$  a.e.

Riemann-Lebesgue Lemma

Fejers Thm

Lemma  $\int_{-\pi}^{\pi} |f(t)| dt < \infty$   
 $\Rightarrow \sigma_n(f(x)) \xrightarrow{n \rightarrow \infty} f(x)$

from Lemma block

Localization principle

Dini's thm

Inner product / Hilbert spaces  
- Cauchy - Schwartz

Linear Approx. Lemma

- Orthonormal Basis Thm  
- Bessel's (Znlegiaty)  
- Parseval's identity

orthonormal basis

$\{e^k\}_{k \in \mathbb{Z}}$  on b  
for  $L_2(\mathbb{T})$

abstract summability thm

- Riesz-Fischer Thm  
- Plancherel Thm  $\|S_n(f)\|_2 \rightarrow \|f\|_2$

Fourier algebra  
 $A(\mathbb{T}) \cong A(\mathbb{S}^1)$

pointwise convergence  
 $S_n(f, t)$ ,  $f$  is piecewise  
boundedly diff.

Gibbs phenomenon

Localisation Thm

Convergence of  
 $S_n(f)$  [on  $f$ ]  
near jump dis continuity