

May 13th, 2008 Random graphs CO 739

$G(n, m)$ $\xrightarrow{\text{smear}}$ $G(n, p)$
 $\xleftarrow{\text{slice}}$

Connection thm:

Hypotheses $p = p(n)$ $pn \rightarrow \infty$ $q = 1-p$, $N = \binom{n}{2}$

Q : a graph property $G_m \in G(n, m)$ $G_p \in G(n, p)$

$X = X(m, n, p) = m - pN + X \sqrt{pqN}$

A : (X is bounded) $\Rightarrow G_m \in Q$ a.s. (depends on p & Q)

[e.g. $p = \frac{2 \log n}{n}$, Q is "G is connected"

A is true]

Conclusion (i) $A \Rightarrow G_p \in Q$ a.s.

(ii) If Q is convex, & $G_p \in Q$ a.s. then A

proof: let $M = |E(G_p)|$

(i) let $\epsilon > 0$ we show $IP(G_p \in Q) > 1 - \epsilon$

for all n sufficiently large.

M is $\text{Bin}(N, p)$, so De Moivre-Laplace says

$\exists C$ st. $IP(M < pN - C \sqrt{pqN}) < \frac{\epsilon}{4}$ (n sufficiently large)

similarly,

$IP(M > pN + C \sqrt{pqN}) < \frac{\epsilon}{4}$ ie. for n large

$IP(X(m, n, p) > C) < \frac{\epsilon}{2}$

Now $IP(G_p \in Q) = \sum_{0 \leq m \leq N} IP(M=m) \cdot IP(G_p \in Q | M=m)$ (law of total prob.)

$\geq \sum_{m: |X(m, n, p)| > C} IP(M=m) \cdot IP(G_p \in Q | M=m)$
 $+ \sum_{m: |X(m, n, p)| \leq C} IP(M=m) \cdot IP(G_p \in Q | M=m)$
 $\geq \sum_{m: |X(m, n, p)| \leq C} IP(M=m) \cdot IP(G_p \in Q | M=m)$
 $\geq \sum_{m: |X(m, n, p)| \leq C} IP(M=m) \cdot IP(G_p \in Q)$
 $\geq IP(X \leq C) \cdot IP(G_p \in Q)$

$$m_0 = m_0(n)$$

$$\frac{\varepsilon}{2} + \sum_{m: |x(m, n, p)| \leq C} P(M=m) P(G_m \in Q)$$

(m_0 minimizes $P(G_m \in Q)$ over $m: |x(m, n, p)| \leq C$)

$$= P(|x(m, n, p)| \leq C) \cdot P(G_{m_0} \in Q) \geq (1 - \frac{\varepsilon}{2}) \cdot (1 - o(n))$$

by property A.

$> 1 - \varepsilon$ for n suff. large. Hence $G_p \in Q$ a.s.

NB similar to one of these steps.

if A_1, \dots, A_k is a partition of event B and Q is a property, $P(Q | B) \leq \max_i P(Q | A_i)$

Lemma: If Q is convex and $m_1 \leq m \leq m_2$, then

$$P(G_m \in Q) \geq P(G_{m_1} \in Q) + P(G_{m_2} \in Q) - 1$$

proof: $P(G_m \in Q) = 1 - P(G_m \notin Q)$

$G(n) = (G_0^*, G_1^*, \dots, G_N^*)$ random graph process

$$P(G_m \in Q) = P(G_m^* \in Q) \geq P(G_{m_1}^* \in Q \wedge G_{m_2}^* \in Q)$$

by convexity

$$= 1 - P(G_{m_1}^* \notin Q \vee G_{m_2}^* \notin Q)$$

$$\geq 1 - P(G_{m_1}^* \notin Q) - P(G_{m_2}^* \notin Q)$$

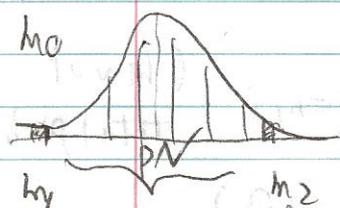
$$= P(G_{m_1}^* \in Q) + P(G_{m_2}^* \in Q) - 1 \quad \square$$

proof of (ii) Assume $G_p \in Q$ a.s. Q convex

Let $C > 0$. Assume $m = m(n)$ is st.

$$|x(m, n, p)| \leq C$$

$$\text{Let } m_1 = \lfloor pN - C\sqrt{pN} \rfloor \quad m_2 = \lceil pN + C\sqrt{pN} \rceil$$



Note $x(m, n, p) = -c + o(1)$

Let m_0 maximize $P(G_m \in Q)$ for $m = m_0 < m_1$

Then $P(G_p \in Q) = T_1 + T_2 + T_3$

where $T_1 = \sum_{m < m_1} \underbrace{P(G_p \in Q / \mu = m)}_{\substack{\mu = m \\ \mu < m_1}} P(\mu = m)$

$T_2 = \sum_{m_1 \leq \mu \leq m_2}$ This is $P(G_m \in Q)$

$T_3 = \sum_{\mu > m_2}$

$T_1 \leq \sum_{m < m_1} P(G_{m_0} \in Q) P(\mu = m) = P(G_{m_0} \in Q) P(\mu < m_1)$
 $= P(G_{m_0} \in Q) [\Phi(-c) + o(1)]$ by DeM-L.

Similarly, $T_3 \leq P(G_{m_3} \in Q) (1 - \Phi(c) + o(1))$

$T_2 \leq P(m_1 \leq \mu \leq m_2) = \Phi(c) - \Phi(-c) + o(1)$

So $P(G_p \in Q) = P(G_{m_0} \in Q)$ $m_0 \leq m \leq m_3$
 $= T_1 + T_2 + T_3$ But by the Lemma.

$(1 - o(1)) \cdot \sup \{ P(G_{m_0} \in Q) \mid P(G_{m_3} \in Q) \geq 1 - o(1) + (1 - o(1)) \}$
 $= 1 - o(1)$

i.e. so $G_m \in Q$ a.a.s \square

Threshold functions.

$f = f(n)$ is a threshold function for Q . if

$$\lim_{n \rightarrow \infty} P(G(n, p) \text{ has } Q) = \begin{cases} 0 & \text{if } p = o(f) \\ 1 & \text{if } p = w(f) \\ & \text{i.e. } f = o(p) \end{cases}$$

e.g. "has a perfect matching" $f = \frac{\log n}{n}$

f is a sharp threshold function for Q if $\forall \epsilon > 0$

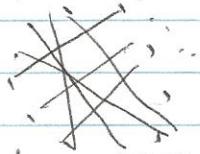
$$\lim_{n \rightarrow \infty} \mathbb{P}(G(n, p) \text{ has } Q) = \begin{cases} 0 & \text{if } p < (1-\epsilon)f(n) \\ 1 & \text{if } p > (1+\epsilon)f(n) \end{cases}$$

$0 \leftrightarrow 1$ still called a threshold.

Let $x(G) =$ no. of isolated vertices.
($i: d(i) = 0$)

$$Q = \begin{cases} \{x > 0\} \\ \{x = 0\} \end{cases}$$

$G(n):$



Indicator RV. X with values 0 & 1.

X indicator for Q if $X = \begin{cases} 0 & \text{if } Q \text{ false} \\ 1 & \text{if } Q \text{ true} \end{cases}$

For such X . $E(X) = \mathbb{P}(X=1)$

If $X =$ no. of isolates then

$$X = \sum_{i=1}^n x_i \quad \text{where } x_i \text{ indicator for } \begin{cases} 0 & \text{if } d(i) \neq 0 \\ 1 & \text{if } d(i) = 0 \end{cases}$$

Linearity of E . If X, Y RV's. a, b constants, then

$$E(aX + bY) = aE(X) + bE(Y)$$

$$E X = \sum_{i=1}^n E x_i = \sum_{i=1}^n \mathbb{P}(x_i = 1)$$

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Indicator $I_A = \begin{cases} 1 & A \text{ hold} \\ 0 & \text{otherwise} \end{cases}$

$$E I_A = \mathbb{P}(I_A = 1) = \mathbb{P}(A) \quad (A \text{ an event})$$

$G(n, p)$ $X =$ no. isolates $= \sum_{i=1}^n x_i$ $x_i = I_{\{d(i)=0\}}$

$$E X = E \sum x_i = \sum_{i=1}^n E x_i = \sum_{i=1}^n \mathbb{P}(d(i)=0)$$

$$= n \mathbb{P}(d(i)=0) = n(1-p)^{n-1}$$



$p \rightarrow 0$ as $n \rightarrow \infty$

$$\log(1+x) = x + o(x^2) \quad (x \rightarrow 0)$$

Let $p = \frac{c \log n}{n}$, Then $E X = n \exp(n(-p + o(p^2)))$ 15.

$$= n \exp(-np + o(np^2))$$

$$= n \cdot e^{-c \log n} \cdot e^{o(1)} \quad \left(\begin{array}{l} \downarrow \\ 0 \end{array} \right) \Rightarrow \frac{c^2 \log^2 n}{n^2} \rightarrow 0$$

$$\sim n^{1-c} \quad \rightarrow = 1 + o(1) \text{ by Taylor}$$

For $c < 1$ $E X \rightarrow \infty$, $c > 1$ $E X \rightarrow 0$
 Threshold $p = o(1) \Rightarrow Q$ false

$p = w(f) \Rightarrow Q$ true a.s.

$$P(X \neq 0) = \frac{1}{\log n}$$

Notice that $E(X) \rightarrow \infty$ | a.s. $X=0$

$$P(X=0) = 1 - \frac{1}{\log n} \rightarrow 1$$

Let X be a non-negative rv. space Ω

Assume $E X \neq 0$ Let $t > 1 > t E X$

$$E X = \sum_{X(\omega) \leq t E X} X(\omega) \cdot P(\omega) + \sum_{X(\omega) > t E X} X(\omega) \cdot P(\omega)$$

$$> t E X \cdot P(X(\omega) > t E X)$$

$$\frac{1}{t} > P(X > t E X) \quad (\text{Markov's Inequality})$$

even for $E X = 0$

$$\text{Also } P(X \geq t E X) \leq \frac{1}{t} \quad (t > 1)$$

Cor. $X \geq 0$ rv. integer
 $P(X \geq 1) \leq E X$

First Moment principle.

$[t = \frac{1}{E X}]$ X integer.

Cor 2. If $X \geq 0$ $E X \rightarrow 0$ then $X=0$ a.s.

So for $c > 1$, $p = \frac{c \log n}{n}$

$X = \#$ isolated vertices
 $= 0$ a.s. (since $E X \rightarrow 0$)

Q: "has no isolates" increasing property.

variance: $E X^2 - (E X)^2 = \sigma^2 = \text{var} X$

$= E (X - E(X))^2$

Let $Y = (X - E X)^2$

$E Y = \sigma^2$ Markov \Rightarrow

$P(Y > S \sigma^2) < \frac{1}{S}$ for $S > 1$

\downarrow

$P((X - E X)^2 > (t \sigma)^2) < \frac{1}{t^2}$ ($s = t^2$)

\hookrightarrow

$P(|X - E X| > t \sigma) < \frac{1}{t^2}$ (Chebyshev's inequality) ($t > 1$)

Cor. Suppose X is RV such that $E X \rightarrow \infty$ and

$E(X(X-1)) \sim (E X)^2 \Rightarrow$ Then $X \neq 0$ a.s.

Proof:

$E(X(X-1)) = E[X^2 - X] = E(X^2) - E X$

since this is $\sim (E X)^2$

$E(X(X-1))$

$E(X^2) - E X = (E X)^2 + o((E X)^2)$

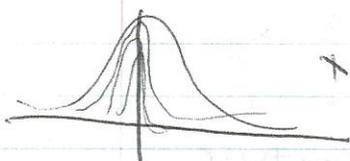
\uparrow $o((E X)^2)$ as $E X \rightarrow \infty$

So $\sigma^2 = E X^2 - (E X)^2$ is $o((E X)^2)$

Put $t = \frac{E X}{\sigma}$ in Chebyshev:

$P(|X - E X| \geq \frac{E X}{\sigma} \cdot \sigma) < \frac{\sigma^2}{(E X)^2} = o(1)$

So $P(X=0) \rightarrow 0$



$E X$ $X = \text{no. of isolates.}$

$X = \sum_{i=1}^n x_i$ — as before

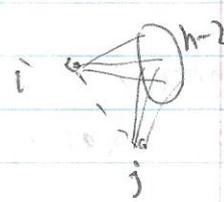
$x_5 \quad x_4$

2nd moment method

$X(X-1) = \#$ ordered pairs of distinct isolates.

$= \sum_{i \neq j} X_i X_j$ indicators i, j both degree 0

$X_2 X_3 = \begin{cases} 1 & \text{if } X_2, X_3 \text{ isolated} \\ 0 & \text{otherwise} \end{cases} = \sum_{i \neq j} P(d(i)=0 \wedge d(j)=0)$



$E(X(X-1)) = n(n-1) P(d(1)=0 \wedge d(2)=0)$
 $= n(n-1) (1-p)^{2n-2} \sim n^2 (1-p)^{2n-2}$ as $p \rightarrow 0$ and $(1-p) \rightarrow 1$.

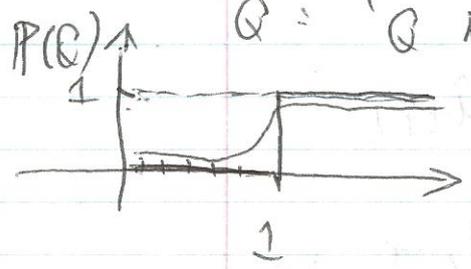
$E X = n(1-p)^{n-1} = (E X)^2$

so $P(X=0) \rightarrow 0$ by 2nd moment method. a.s. $X \geq 1$.

provided $E(X) \rightarrow \infty$ ($c < 1$ $p = c \frac{\log n}{n}$)

so for $c < 1$ $G \in \mathcal{G}(n, p)$ a.s. Q is false

$Q = "Q \text{ has no isolates}"$



limit \rightarrow step func. So $\frac{\log n}{n}$ is a sharp threshold func.

for Q :

suppose Z rv with $P(Z=i) = \frac{e^{-\lambda} \lambda^i}{i!}$

then Z has Poisson distribution with mean λ . fixed dist.

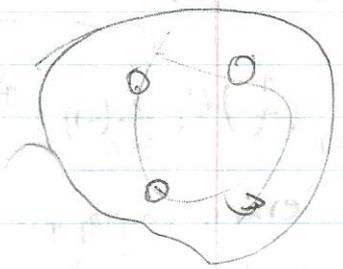
$Z \stackrel{d}{\sim} P_0(\lambda)$

let X_1, X_2, \dots and Z be integer rv's such that $\forall k \in \mathbb{Z} P(X_n = k) \rightarrow P(Z = k)$ as $n \rightarrow \infty$

$X_n = X_{1/n} + \dots + X_{n/n}$ indep. indicators.

$E X \rightarrow \lambda$, Then $X_n \xrightarrow{d} P_0(\lambda)$ $P(Z=k)$ as $n \rightarrow \infty$

Then we write $X_n \xrightarrow{d} Z$.



If $Z \stackrel{d}{=} P_0(x)$ we say X is asymptotically Poisson with mean λ .

$$P = \frac{\log n}{n} + f(n\lambda) \quad X = \# \text{ of isolates} \quad \text{get } X \xrightarrow{d} IP_0(\lambda)$$

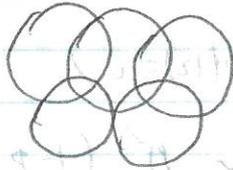
$$g(n) = \begin{cases} a & n \text{ even} \\ b & n \text{ odd} \end{cases} \quad Z \sim P(\lambda)$$

can have $P(X=0) = g(n) + o(1)$.

May 21th 2008

$G(n, p)$

Inclusion-Exclusion



$$A_1, \dots, A_2, \dots, A_m \subseteq S \text{ (finite)}$$

$e_i = \#$ of elements of S in exactly i of the A_j

then "have" $a_j = \sum_{1 \leq i_1 < \dots < i_j \leq m} |A_{i_1} \cap \dots \cap A_{i_j}|$

Then $a_j = \sum_{i=j}^m \binom{i}{j} e_i$ (*) $\binom{3}{2}$

Thm (Bonferroni's inequalities)

Let e_1, \dots, e_m be non-negative reals.

Let $t \geq 0$ then for all $u \geq 0$

$$\left| e - \sum_{j=0}^u (-1)^j \binom{t+j}{j} a_{t+j} \right| \leq \binom{t+u}{u} a_{t+u}$$

where the a_j 's are given by (*)

proof:

$$\sum_{j=0}^u (-1)^j \binom{t+j}{j} a_{t+j} = \sum_{j=0}^u (-1)^j \binom{t+j}{j} \sum_{i=t+j}^m \binom{i}{t+j} e_i$$

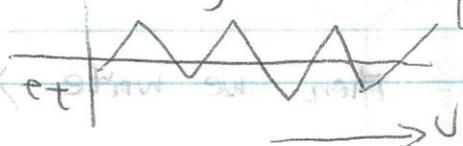
$$= \sum_{i=t}^m \sum_{j=0}^{\min(u, i-t)} (-1)^j \frac{i!}{j!(t+j)!(i-t-j)!} e_i$$

$$= \sum_{i=t}^m \sum_{j=0}^u (-1)^j \binom{i}{t} \binom{i-t}{j} e_i$$

$$= \sum_{i=t}^m \binom{i}{t} e_i \sum_{j=0}^u (-1)^j \binom{i-t}{j} = e t + \sum_{j=t+1}^m \binom{j}{t} e_j \sum_{i=0}^u (-1)^i \binom{j-t}{i}$$

$$= e t + \sum_{i=t+1}^u \binom{i}{t} e_i (-1)^u \binom{i-t-1}{u-i}$$

alternating "errors" in partial sums.



$$P(X_n = j) \rightarrow \frac{e^{-\lambda} \lambda^j}{j!}$$

↑
fixed

e.g. Bin(n, p) with np = λ ← fixed

$$\binom{n}{j} p^j (1-p)^{n-j} \sim \frac{n^j}{j!} \left(\frac{\lambda}{n}\right)^j \left(1 - \frac{\lambda}{n}\right)^{n-j}$$

$$P(X_n = j) = \frac{e^{-\lambda} \lambda^j}{j!} + o(1)$$

Method of moments

$$[X]_j = X(X-1)\dots(X-j+1) \quad \frac{[X]_j}{j!} = \binom{X}{j}$$

Thm. suppose X_n is a r.v. bounded ≥ 0 for each n and $\lambda = \lambda(n)$ is bounded, such that $E([X_n]_j) = \lambda^j + o(1)$ for all fixed int. $j \geq 0$

Then for such j , $P(X_n = j) = \frac{e^{-\lambda} \lambda^j}{j!} + o(1)$

proof: Define $e_i = P(X_n = i)$

Then $a_j = \sum_{i=j}^m \binom{i}{j} e_i = \sum_{i=j}^m \binom{i}{j} \frac{[X_n]_j}{j!} e_i$ Assume $X_n \leq m$

$$= \frac{1}{j!} E([X_n]_j) = \frac{1}{j!} \lambda^j + o(1)$$

By Bernoulli,
for fixed U

$$\left| e^t - \sum_{r=0}^U (-1)^j \binom{t+j}{r} \frac{\lambda^{t+j}}{(t+j)!} + o(1) \right|$$

$$\leq \binom{t+U}{U} \frac{\lambda^{t+U}}{(t+U)!} + o(1)$$

ignore

$$= O\left(\frac{t^U}{U!} \cdot \frac{\lambda^U}{(t+U)!}\right) \quad \text{for large } U.$$

which is $< \varepsilon$ for U suff. large (fixed t)

For such U ,

the sum is $\sum_{r=0}^{U-1} (-1)^j \binom{t+j}{r} \frac{\lambda^t \lambda^j}{(t+j)!} = \sum_{r=0}^{U-1} \left(\frac{(-1)^j \lambda^j}{j!} \right)$

$$= e^{-\lambda} + \varepsilon \quad \text{for } U \text{ suff. large}$$

Hilroy

So let $\left| e^{-\lambda} \frac{\lambda^t}{t!} - \frac{e^{-\lambda} \lambda^t}{t!} \right| \leq \epsilon + o(1)$ holds $\forall \epsilon > 0$

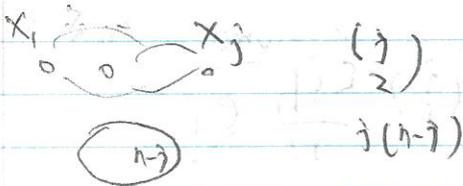
so $e^{-\lambda} \frac{\lambda^t}{t!} = \frac{e^{-\lambda} \lambda^t}{t!} + o(1)$
 $g(n) \rightarrow 0$
 $n \rightarrow \infty$

Let $X_n = \#$ of isolates in $G(n, p)$

$X_i = \mathbb{I}_{\{d(x_i) = 0\}}$ $X = \sum_{i=1}^n X_i$

Then $[X]_j = X(X-1)\dots(X-j+1) = \#$ ordered isolated vertices

$= \sum_{i_1, i_2, \dots, i_j \text{ all different}} X_{i_1} X_{i_2} \dots X_{i_j} \Rightarrow \mathbb{E}[X]_j = \mathbb{E} \sum X_{i_1} \dots X_{i_j}$
 $= [n]_j \cdot \mathbb{E} X_1 \dots X_j$



$= [n]_j \mathbb{P}(X_1=1, \dots, X_j=1)$
 $= [n]_j (1-p)^{\binom{j}{2} + j(n-j)}$
 $\sim n^j (1-p)^{\binom{j}{2} + j(n-j)}$

Take $p = \frac{\log n + x}{n} \rightarrow 0$ i fixed

$\mathbb{E}[X]_j \sim n^j (1-p)^{\binom{j}{2} + j(n-j)} \sim \lambda^j$ where $\lambda = n(1-p)^{n-1}$

so $\lambda = n \exp(n \log(1-p))$

$= n \exp(n(-p + o(p^2))) = n \exp\left(n\left(-\frac{\log n}{n} - \frac{x}{n}\right) + o(1)\right)$

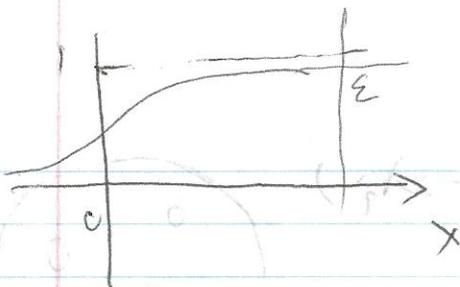
$= n \left(\frac{1}{n} e^{-x}\right) e^{o(1)} \rightarrow e^{-x}$

so method of moments \Rightarrow

$X \xrightarrow{d} \text{IP}_d(e^{-x})$

eg $\mathbb{P}(X=0) \rightarrow e^{-e^{-x}}$

no isolates



Cor. If $\frac{p \log n + X(n)}{n}$ then (1) if $X(n) \rightarrow \infty$, $G(n, p)$ a.a.s. has no isolates.

(2) $X(n) \rightarrow -\infty$, $G(n, p)$ a.a.s. has an isolate.

$\frac{n}{2}$ even
 $G(n, p)$ $p = \frac{c}{n}$, let $X =$ no. perfect matchings.
 $E X = \sum_i E X_i$ where X_1, \dots, X_t are indicators for the event s

that the various perfect matchings are present

$$t = \frac{n!}{(\frac{n}{2})! 2^{n/2}}$$

$$E X = \frac{n!}{(\frac{n}{2})! 2^{n/2}} \cdot p^{n/2} \sim \frac{(n/e)^n \sqrt{2\pi n}}{(\frac{n}{2})! 2^{n/2}} \left(\frac{c}{n}\right)^{n/2}$$

stirling.

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

$$= \sqrt{2} \left(\frac{n}{e} \times \frac{c}{n}\right)^{n/2}$$

$$= \sqrt{2} \cdot \left(\frac{c}{e}\right)^{n/2} \rightarrow \infty \text{ for } c >$$

eg. $c=3$ But a.a.s. have many isolates.
 \Rightarrow no p.m.

May 22nd, 2008

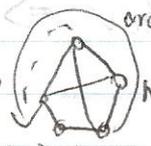
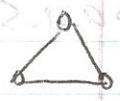
$G(n, p)$ Let X_i be the number of i -cycles



$E(\text{length of 1st cycle}) \hat{=} n^{1/6}$

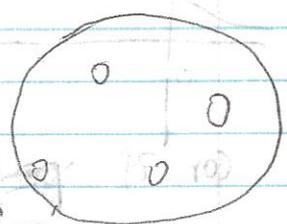
If $w(n) \rightarrow \infty$ then a.a.s. length $< w(n)$

$$E(X_i) = \sum_{\text{ordered } i\text{-cycles}} p^i = \frac{[n]_i}{2^i} p^i \sim \frac{(pn)^i}{2^i} \quad (i \text{ will be fixed})$$

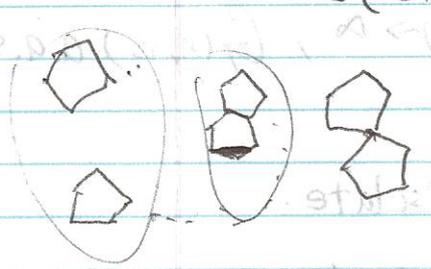


$(n-i)! n$ 2^i # auto-morph.

11. For $p = \frac{c}{n}$ $E[X_i] \sim \frac{c^i}{2i} (=:\lambda_i)$



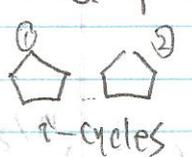
Consider $E([X_i]_2) = \sum_{\substack{\text{ordered pairs } C_1, C_2 \\ i\text{-cycles in } K_n}} P(C_1, C_2 \subseteq G) \in G(n, p)$



$= \Sigma_1 + \Sigma_2$
 where $\Sigma_1 = \sum_{\substack{C_1, C_2 \subseteq K_n \\ (i\text{-cycles}) \\ V(C_1) \cap V(C_2) = \emptyset}} P(C_1 \cup C_2 \subseteq G)$

$\Sigma_2 =$ the rest!

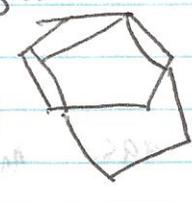
No. of summands in Σ_1 is: $\frac{[n]_{2i}}{(2i)^2} \sim \frac{n^2}{(2i)^2}$ Each term is p^{2i}



so $\Sigma_1 \sim \frac{n^{2i} p^{2i}}{(2i)^2} = \frac{c^{2i}}{(2i)^2} = \lambda_i^2$

$\Sigma_2 = \sum_{i < j \leq 2i} \sum_{H \in \mathcal{H}(i, j)} (\# \text{ copies of } H \text{ in } K_n) \times p^{|E(H)|}$

H a union of two i -cycles with vertex intersection and ν vertices and μ edges (red & blue)

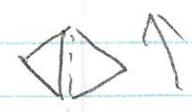


≥ 2 cycles, connected. Let $\nu = \#$ vertices, $\mu = \#$ edges.

Then $\mu > \nu$ (Remove 2 edges & keep connected leaves at least $\nu - 1$ edges)

$\leq \sum_{\nu < \mu \leq 2i} \sum_{H \in \mathcal{H}(\nu, \mu)} n^\nu \cdot p^\mu = \left(\frac{c}{n}\right)^\mu$

overcount $O(n^{-1}) = O(n^{-1})$



so $E([X_i]_2) \sim \lambda_i^2$

If $c \rightarrow \infty$ Chebyshev \Rightarrow a.a.s. $\lambda_i > 0$

if C constant ($p = \frac{C}{n}$)

Exercise $E([X_i]_t) \sim \lambda_i t$ for all fixed t .

Then by method of moments, $X_i \xrightarrow{d} \text{Po}(\lambda_i)$ ($\lambda_i = \frac{C^i}{i!}$)

$$\leftarrow \begin{matrix} n-1 \\ E(\deg(v)) = \frac{n-1}{n} \cdot C \rightarrow C \end{matrix}$$

e.g. $P(X_i=0) \rightarrow e^{-\lambda_i}$

Defn. X_1, X_2, \dots, X_r are nonneg. integer random variables with

We say X_1, \dots, X_r are asymptotically indept. Poisson with means

$\lambda_1, \dots, \lambda_r$ if $(\forall \text{ fixed } t_1, \dots, t_r \geq 0 \in \mathbb{Z})$

$$P(X_1=t_1 \wedge \dots \wedge X_r=t_r) \rightarrow \prod_{i=1}^r \frac{e^{-\lambda_i} \lambda_i^{t_i}}{t_i!}$$

Thm (Generalised method of moments)

If X_1, \dots, X_r are non-neg. int. rv's and for all fixed non-neg integers u_1, \dots, u_r ,

$$E([X_1]_{u_1} [X_2]_{u_2} \dots [X_r]_{u_r}) \sim \prod_{i=1}^r \lambda_i^{u_i} \text{ for fixed } \lambda_1, \dots, \lambda_r.$$

then X_1, \dots, X_r are asymptotically Poisson with means $\lambda_1, \dots, \lambda_r$.

Exercise (*) holds for X_3, \dots, X_r where $X_i = \#$ i -cycles in $G(n, p)$

$$p = \frac{C}{n}$$

so X_3, \dots, X_r are as. indept. Poisson. $p = \frac{C}{n} \quad \frac{C^i}{i!} = \lambda_i$

$C \rightarrow 0$ as no 1-cycles

$C \rightarrow \infty$ $\dots \geq 1$ 1-cycles (Chebyshev)

$\frac{1}{n}$ is threshold function for "has an i -cycle" in $G(n, p)$

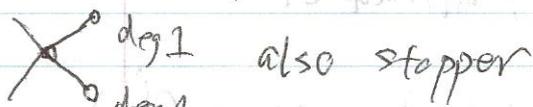
$Z_n G(n, m)$, $m = \frac{1}{2} Cn$, can compute moments \rightarrow similar result.

$$p = \frac{C}{n}$$

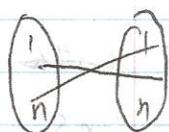
$$p = \frac{\log n}{n} + \frac{X}{n}$$

Ex - What is $\mathbb{P}(G \text{ has a perfect matching?})$

isolated vertex is a "stopper" for a perfect matching.



cherry Let $G(n, n, p)$ be the random bipartite graph with vertex set $\{(1, i), (2, i) : i \in [n]\}$ with edges "across" each having prob p . (independently)



Put $p = \frac{\log n}{n} + \frac{x}{n} \leftarrow$ bounded

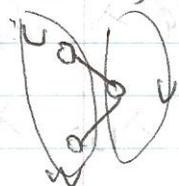
$X =$ no. isolated vertices

$$\mathbb{E} X = 2n(1-p)^n = 2n \exp(n(-p + o(p^2)))$$

$$= 2n \exp(-np + o(n)) = 2n e^{-\log n} e^{-x} = 2e^{-x}$$

Exercise. X is as Poisson.

A cherry of G is a subgraph uvw with $uv, vw \in E(G)$ and



$d(u) = d(w) = 1$

Take $p > \frac{1}{2} \frac{\log n}{n}, < \frac{2 \log n}{n}$

$\left(p = \frac{c(n) \log n}{n} \right)$

Let $Y =$ # cherries in $G(n, n, p)$

$$\mathbb{E} Y = (2n) \binom{n}{2} p^2 (1-p)^{2n-2} = O(n^3) O\left(\frac{\log^2 n}{n^2}\right) \exp(-(2n-2)p)$$

unordered cherry

$\boxed{1-p < e^{-p}}$
 $0 < p < 1$

$= O(n \log^2 n) e^{-2c \log n}$

$= O(n^{1-2c} \log^2 n)$

$= O(1)$ for $c > \frac{1}{2} + \epsilon$ ($\epsilon > 0$)

By Markov, a.s. $Y=0$ for

$\frac{(\log n) (\frac{1}{2} + \epsilon)}{n} < p < \frac{2 \log n}{n}$

May 27th, 2008

CO 739 Random Graph

15

Aside $\binom{n}{k} \sim \frac{n^k}{k!}$

Lemma for $n \geq 1, 0 \leq k \leq n$

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

proof: Exercise 2 ways

1) Stirling's formula with error

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} (1 + \delta) \quad 0 \leq \delta \leq \frac{1}{12n}$$

(ii) induction on k

Hall's thm: If G is bipartite with bipartition (V_1, V_2) $|V_1| = |V_2| = n$
then G has a perfect matching iff

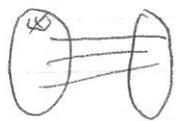
$$|N(S)| \geq |S| \quad \forall S \subseteq V_1$$

Assume there is a "bad" set S , i.e. $S \subseteq V_1$ or V_2 such that

$$|N(S)| < |S|$$

Let S be chosen to minimize $|S|$ (amongst bad S)

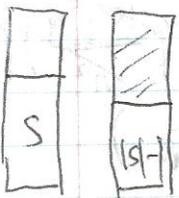
Then (i) $|N(S)| = |S| - 1$ (since if $|N(S)| \leq |S| - 2$



then $S \setminus \{v\}$ is also bad for any $v \in S$
 $\deg(v) = 0$ contradicting minimality of S .

$$(ii) |S| \leq \lceil \frac{n}{2} \rceil$$

since otherwise, assume $S \subseteq V_1$



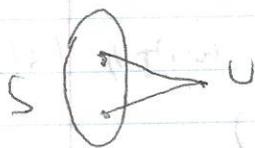
$V_2 \setminus N(S)$ is also bad, no edges go from it to S .

$$|N(S)| \text{ and } |V_2 \setminus N(S)| = n - |S| + 1 \leq (n - (\lceil \frac{n}{2} \rceil + 1) + 1)$$

Contradicts minimality of S .

$$= n - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor \leq |S|$$

(iii) if $u \in N(S)$, then u has at least 2 neighbours in S



(since if $\circ \rightarrow u$, then $S \setminus \{v\}$ is bad)

let x_k be the number of bad sets S satisfying

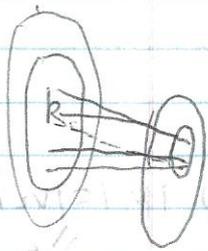
(i) (ii) (iii) with $|S| = k$

By (ii) $k \leq \lceil \frac{n}{2} \rceil$ (o.w. $X_k = 0$)

X_1 = no. of isolates

X_2 = no. of cherries

$$EX_k \leq 2 \binom{n}{k} \binom{n}{k-1} \binom{k}{2}^{k-1} \cdot P^{2(k-1)} \cdot (1-P)^{k(n-k+1)}$$



$v_1, v_2 \in V(S)$ choose $v_1, v_2 \in S$ edges present \swarrow no edges from S to $V_2 \setminus M(S)$

[sum of indicators, each for an event that specifies $S, M(S), 2$ edges from each $v \in V(S)$ no edges S to $V_2 \setminus M(S)$]

These indicators over count the bad sets sat. (i)-(iii)

$$so EX_k \leq 2 \left(\frac{en}{k}\right)^k \left(\frac{en}{k}\right)^{k-1} \left(\frac{k^2}{2}\right)^{k-1} \cdot P^{2k-2} \cdot \exp[-P \cdot k \left(\frac{n}{2}\right)]$$

$\binom{n}{k-1} \leq \binom{n}{k}$ as $k \leq \lceil \frac{n}{2} \rceil$

Take $\epsilon > 0$, $\frac{(1-\epsilon) \log n}{n} < P < \frac{(1+\epsilon) \log n}{n}$

Then $EX_k \leq 2 \left(\frac{e^2 n^2}{k^2} \cdot \frac{k^2}{2} \cdot P^2 \cdot e^{-Pn/2}\right)^k P^{-2} \leq O\left(\frac{\log^2 n}{n^2}\right) \left(\frac{e^2 (1+\epsilon)^2 \log^2 n}{n^2}\right)$

$$e^{-Pn/2} = e^{-((1-\epsilon)/2) \log n} = n^{-\frac{1-\epsilon}{2}} \cdot \frac{1}{n} \left(\frac{k^k}{(k-1)^{k-1}}\right) \leq \left(\frac{1}{\sqrt{n}}\right)^k$$

For $k \geq 3$, this $O(1)$ ratio $\frac{B_{k+1}}{B_k} = O(1)$ ($\epsilon < \frac{1}{3}$) ($< \frac{1}{2}$ for suff. large e) for $k \geq 3$ ($\epsilon < \frac{1}{3}$)

So $\sum_{k=3}^{\lceil \frac{n}{2} \rceil} EX_k = o(1)$ ($\epsilon < \frac{1}{3}$)

So a.a.s. $G_{(n, n, p)}$ has no bad sets S with $|S| \geq 3$ ($\epsilon < \frac{1}{3}$)

so for $p = \frac{\log n + X}{n}$

we have $X_2 = 0$ a.s.

$X_k = 0 \quad \forall k \geq 3$ a.s.

So $G \in G(n, n, p)$ a.s. has a perfect matching iff $X_1 = 0$

$(|X| < \frac{1}{4} \log n)$

So $P(G(n, n, p))$ has a perfect matching

$$\rightarrow \begin{cases} 0 & X \rightarrow \infty \\ e^{-2e^{-X}} & X \rightarrow c \\ 1 & X \rightarrow -\infty \end{cases}$$

First for $X \rightarrow \infty$ but $|X| < \frac{1}{4} \log n$, then for larger X by monotonicity)

Define $G(n, n, m)$ to be $G(n, n, p)$ restricted to graphs with m edges.

Connection theorem holds also between $G(n, n, m)$ & $G(n, n, p)$.

Let N denotes N^2 now. PN

For $p = \frac{\log n}{n} - \frac{\log \log n}{n}$ ($N = n^2$)

Consider a.s. $G(n, n, p)$ has no p.m.

so if $m = n(\log n - \log \log n) + X$

a.s. $G(n, n, m)$ has no p.m.

(X is bounded)

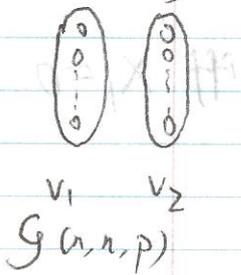
for $m_1 = \lfloor n \log n - n \log \log n \rfloor$ a.s. no p.m.

& for $m_2 = \lfloor n \log n + n \log \log n \rfloor$ a.s. has p.m.

in $G(n, n, m_i)$ $i=1, 2$)

Define $G^{Bip}(n)$ to be the bipartite random graph process
 — analogue of $G(n)$

May 29th, 2008.



"bad" set S if $S \subseteq V_1$ or V_2 $|N(S)| < |S|$

$X_k = \text{No. of bad set } S \text{ of size } k, (k \geq 2)$

$$\mathbb{E} X_k \leq O(n^k) \left(\frac{O(1) \log^2 n}{n^{(1-\epsilon)/2}} \right)^k$$

where $\mathbb{P}\left(\frac{(1-\epsilon) \log n}{n}, \frac{(1+\epsilon) \log n}{n} \right)$

eg. $\epsilon = \frac{1}{4}$

$\mathbb{E} X_k \rightarrow 0$
 $k \geq 2$

$$\mathbb{P}(\text{No. of isolates}) \rightarrow \begin{cases} 0 & p = \frac{\log n}{n} - \frac{w(1)}{n} \\ 1 & p = \frac{\log n}{n} + \frac{w(1)}{n} \end{cases}$$

$G(n, n, m)$ · m edges.

To make life easy: Expansion Lemma:

Let Q be a graph property, let \mathbb{P}_p denote prob. in $G(n, n, p)$

\mathbb{P}_m in $G(n, n, m)$, then for $p = \frac{m}{n^2} = N$

$$\mathbb{P}_m(Q) = \frac{\mathbb{P}_p(Q)}{\mathbb{P}_p(\text{no isolates})}$$

[Or \mathbb{P}_p in $G(n, p)$, \mathbb{P}_m in $G(n, m)$]

$N = n^2$ for $G(n, n, p)$ $\binom{n}{2}$ for $G(n, p)$

proof: $\mathbb{P}_m(Q) = \mathbb{P}_p(Q | E_m)$ where $E_m = \text{" } G \text{ has } m \text{ edges"}$

$$\text{prob.} = \frac{\mathbb{P}_p(Q \cap E_m)}{\mathbb{P}_p(E_m)} = \frac{\mathbb{P}_p(Q)}{\mathbb{P}_p(E_m)}$$

$$\text{xbw } \mathbb{P}_p(E_m) = p^m (1-p)^{N-m} \binom{N}{m} = p^m (1-p)^{N-m} \frac{N!}{(N-m)! m!}$$

$$= \Theta(1) \left(\frac{m}{N} \right)^m \left(\frac{N-m}{N} \right)^{N-m} \left(\frac{N}{e} \right)^N \sqrt{2\pi N}$$

$$\left(\frac{N-m}{e} \right)^{N-m} \sqrt{2\pi(N-m)} \times \left(\frac{m}{e} \right)^m \sqrt{2\pi m}$$

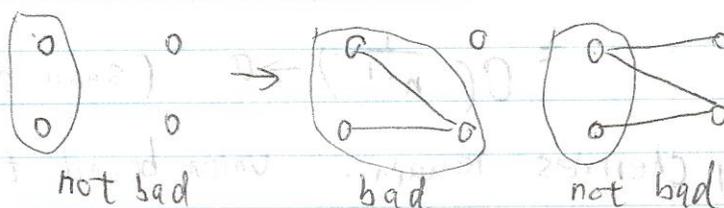
$$= \Theta(1) \frac{1}{N \binom{N}{N-m} \binom{N}{m} \frac{1}{N}} = \Theta(1) \frac{1}{N^2 p(1-p)^2 N}$$

Theorem In $G^{\text{Bip}}(n) = (G_0, G_1, \dots, G_m)$ a.s. the first m with $\delta(G_m) > 0$ is such that G_m has a perfect matching. 19.

let $m_1 = \lceil n \log n - n \log \log n \rceil$ $m_2 = \lfloor n \log n + n \log \log n \rfloor$
 Connection theorem uses $p = \frac{m}{N}$ ($p = \frac{\log n}{n} - \frac{\log \log n}{n} + O(\frac{1}{n^2})$)

a.s. isolates in $G(n, n, p)$
 so a.s. in $G(n, n, m_1) \ni$ isolates.

"No bad sets" not convex
 ($k \geq 2$)



But we do have for $m \in [m_1, m_2]$

put $p = \frac{m}{N}$ so $p \in \left[\frac{\log n}{n} - \frac{\log \log n}{n}, \frac{\log n}{n} + \frac{\log \log n}{n} \right]$
 can take ε arbitrarily small.

$\mathbb{P}_p(X_k > 0) \leq \mathbb{E}_p X_k = O(n^k) \left(\frac{O(1) \log^2 n}{n^{4.5k}} \right)^k$ (by 1st moment principle)

so $\mathbb{P}_m(X_k > 0) = O(n^k \sqrt{n \log n}) \left(\frac{1}{n^{4k}} \right) := B_k$
 $\mathbb{P}(t-p)N = O\left(\frac{\log n}{n} \cdot n^2\right)$

claim a.s. in $G(n, n, m)$ $X_k = 0 \quad \forall k \geq 2$ and all $m \in [m_1, m_2]$
 This implies the theorem because a.s. G_{m_1} has isolates,

G_{m_2} has no isolates, & any graph with no bad set of size ≥ 2 has pm. iff no isolates.

proof of claim: (i) \mathbb{P} (at least one of G_{m_1}, \dots, G_{m_2} has a bad set of size ≥ 7) by union bound, is at most $(m_2 - m_1 + 1)$

$\sum_{k \geq 7} B_k = O(n \log \log n) B_7 = O\left(\frac{(\log n)^{O(1)}}{n^3}\right) = O(n \log \log n)$

(ii) for $k \in \{3, 4, 5, 6\}$

Redo the bound on $\mathbb{E}X_k$

We used $(1-p)^{k(n-b+1)} \leq (1-p)^{kn/2}$ which gave $\frac{1}{\log n^{(1-\epsilon)k/2}}$

But for k bounded, it's $O((1-p)^{kn})$

So for k bounded $\mathbb{P}_m(X_k > 0) = O(n^{3/2} \log n) \left(\frac{O(1) (\log^2 n)^k}{n^{9k}} \right)$

So $\mathbb{P}(\exists m \in [m_1, m_2])$ with such a bad set
 $= O\left(\frac{1}{n^4}\right) \rightarrow 0$ (same argument)

Only cherries remain. Union bound too weak.

Recall $\mathbb{E}(\# \text{ of cherries}) = O(n^3 \frac{\log^2 n}{n^2} \cdot (\frac{\log n}{n} - \frac{\log \log n}{n})^2) + O\left(\frac{1}{n^2}\right)$

So by Expansion lemma, $= O\left(\frac{1}{n} (\log n)^4\right)$

as $G_{n, n, m}$ has no cherries.

Let Y_m be no of cherries created in adding a random edge to G_m to get G_{m+1} .

$\mathbb{E}Y_m = \frac{\sum \text{deg}_i^2}{n(n-1)} \cdot \frac{1}{\binom{N}{m}} \cdot \frac{1}{N-m}$

deg 1
deg 0
w *v*
add vw.
prob(edge vw added)

$= O(n) \frac{[n-2m]_{m-1}}{(m-1)!} \cdot \frac{[n]_m}{m!} \cdot \frac{1}{N-2m} \sim n^2$

$= O(nm) \frac{[N-2m]_m}{[N]_m}$

$$1-x \leq e^{-x} \quad (x < 1)$$

$$\frac{[N-2m]_m}{[N]_m} = \prod_{i=0}^{m-1} \frac{N-2m-i}{N-i} = \prod_{i=0}^{m-1} \left(1 - \frac{2m}{N-i}\right)$$

$$\leq \exp\left(-\sum_{i=0}^{m-1} \frac{2m}{N-i}\right) = \exp\left(-\sum_{i=0}^{m-1} \frac{2m}{n^2} \left(1 + O\left(\frac{m}{n^2}\right)\right)\right) = n^{-2} \left(1 + O\left(\frac{m}{n^2}\right)\right)$$

$$= \exp\left(\frac{m}{n} \log\left(\frac{1}{1 + O\left(\frac{m}{n^2}\right)}\right) + O\left(\frac{m^2}{n^3}\right)\right)$$

$$\approx e^{-2m/n} = e^{-2 \log n + O(\log \log n)} \rightarrow c$$

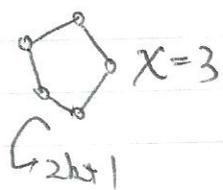
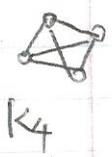
$$= n^{-2} (\log n)^{O(1)}$$

so $E Y_m = O((\log n)^{O(1)} n^{-2})$

so Expected # cherries created is $(m_2 - m_1 + 1) O((\log n)^{O(1)} n^{-2}) = O(1)$

June 2nd, 2008 0739 Random Graphs

$\chi(G)$: chromatic no. of G .



$$\chi \geq \max\{t : K_t \subseteq G\}$$

Let X_t = no. of copies of K_t in $G \in \mathcal{G}(n, p)$.

$$E X_t = \binom{n}{t} p^{\binom{t}{2}}$$

for $p=1/2$, it's $\binom{n}{t} 2^{-\binom{t}{2}} \sim \frac{n^t}{t!} 2^{-t(t-1)/2}$

Fact: If $t = o(n^{1/2})$

$$\binom{n}{t} \sim \frac{n^t}{t!}$$

stirling $\frac{n^t}{t!} \sim \frac{n^t}{(e/t)^t \sqrt{2\pi t}} 2^{t(t-1)/2}$

for $t = \lfloor c \log_2 n \rfloor = O(n)$

$$\leq \left(\frac{en\sqrt{2}}{t 2^{t/2}}\right)^t < \left(\frac{cn}{t}\right)^t \rightarrow 0 \text{ if } c \geq 2$$

$$2^t \geq n^c$$

Exercise show $X_t \geq 1$ a.a.s for $c < 2$ (const)

(or $\chi(G) \geq (2-\epsilon) \log_2 n$ a.a.s. for $G \in \mathcal{G}(n, p=1/2)$)

A martingale is the same but with " $= X_t$ ".
supermartingale inequality

If $X_0 = 0, X_1, \dots$ is a supermartingale with Y_0, Y_1, \dots

and $|X_t - X_{t-1}| \leq C_t$ for all $t \geq 1$
"bounded differences"

then $P(X_t \geq \alpha) \leq e^{-\frac{\alpha^2}{\sum_{i=1}^t C_i^2}}$ (any $\alpha > 0$)

proof: let $h > 0$ (choose h later)

$$P(X_t \geq \alpha) = P(e^{hX_t} \geq e^{h\alpha})$$

$$\leq \frac{E(e^{hX_t})}{e^{h\alpha}} \quad \text{by Markov}$$

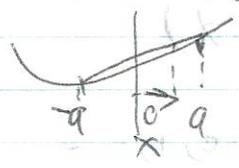
$$E(e^{hX_t}) = E(e^{hX_{t-1}} e^{h(X_t - X_{t-1})})$$

by Tower = $E(E(e^{hX_{t-1}} e^{h(X_t - X_{t-1})} | Y_0, \dots, Y_{t-1}))$

$$= E(e^{hX_{t-1}} \cdot E(e^{h(X_t - X_{t-1})} | Y_0, \dots, Y_{t-1}))$$

by (*) X_{t-1} is determined by Y_0, \dots, Y_{t-1} .

Now e^x is convex



For $x \in [-a, a]$. $e^x \leq \frac{e^a + e^{-a}}{2} + x \frac{e^a - e^{-a}}{2a}$

so $E(e^{h(X_t - X_{t-1})} | Y_0, \dots, Y_{t-1})$

$$\leq E\left(\frac{e^{hC_t} + e^{-hC_t}}{2} + h(X_t - X_{t-1}) \mid h(X_t - X_{t-1}) \leq hC_t\right)$$

$$\leq \frac{e^{hC_t} + e^{-hC_t}}{2} = \cosh(hC_t) \text{ slope}$$

$$E(X_t - X_{t-1}) \leq 0 \quad \text{by supermartingale}$$

ES.

$$\cosh(x) = \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!} \leq \sum_{i=0}^{\infty} \frac{x^{2i}}{2^i i!} = e^{x^2/2}$$

$$\text{So } E[e^{hX_t}] \leq E[e^{hX_{t-1}} \cdot e^{h^2 C_t^2 / 2}] = e^{h^2 C_t^2 / 2} E[e^{hX_{t-1}}]$$

$$\leq e^{\sum_{i=1}^t h^2 C_i^2 / 2} \text{ by induction.}$$

$$\text{So } \Rightarrow P(X_t \geq \alpha) \leq e^{-h\alpha + \frac{h^2}{2} \sum C_i^2}$$

Put $h = \frac{\alpha}{\sum C_i^2}$ for max.

$$P(X_t \geq \alpha) \leq e^{-\frac{1}{2} \alpha^2 / \sum C_i^2}$$

Cor. (Azuma's inequality)

If X_0, X_1, \dots is a martingale (wrt. any \mathcal{Y}_0, \dots)
with $|X_t - X_{t-1}| \leq C_t \quad \forall t$ (C_t constants)

$$P(|X_t - X_0| \geq \alpha) \leq 2e^{-\alpha^2 / 2 \sum_{i=1}^t C_i^2}$$

for all $t \geq 0$ and $\alpha \geq 0$.

Proof: Sequence $(X_t - X_0)_{t=0,1,\dots}$ is a martingale.

so lemma bounds $P(X_t - X_0 \geq \alpha)$

and apply to $(-(X_t - X_0))_{t=0,1,\dots}$

bounds $P(X_t - X_0 \leq -\alpha)$

so \square

June 10th, 2008 CO739 Random Graph

Martingale

X_0, X_1, \dots is martingale wrt $\mathcal{Y}_0, \mathcal{Y}_1, \dots$
if $E(X_{t+1} | \mathcal{Y}_0, \dots, \mathcal{Y}_t) = X_t$

Azuma's Inequality. If $|X_t - X_{t-1}| \leq C_t \quad \forall t$, then

$$P(X_n - EX_n \geq \alpha) \leq 2e^{-\alpha^2 / 2 \sum_{t=1}^n C_t^2}$$

$$P(X_n - EX_n \leq -\alpha) \leq e^{-\dots}$$

$$P(X_n - EX_n \leq -\alpha) \leq e^{-\dots}$$

Special Cases

If Y_t determines Y_0, \dots, Y_{t-1} and $E(X_{t+1} | Y_t) = X_t$
then $(X_t)_{t \geq 0}$ is a martingale wrt $(Y_t)_{t \geq 0}$

Sometimes $E(X_{t+1} | Y_0, \dots, Y_t) = E(X_{t+1} | Y_t)$

& $E(X_{t+1} | Y_t) = X_t$ this holds and it does not necessarily

imply $E(X_{t+1} | Y_0, \dots, Y_t) = E(X_{t+1} | Y_t)$ is true.

Example, let $Z_i \stackrel{\text{def}}{=} \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases}$ — indept.

put $X_t = \sum_{i=1}^t Z_i$ $X_t \stackrel{d}{\sim} \text{Bin}(t, p)$

put $W_t = X_t - tp = \sum_{i=1}^t (Z_i - p)$ $E(X_t) = E(Z_{t+1})$

$$E(W_{t+1} | Z_1, \dots, Z_t) = E(X_{t+1} - tp - p | Z_1, \dots, Z_t)$$
$$= E(X_t - tp + Z_{t+1} - p | Z_1, \dots, Z_t) = E(W_t + Z_{t+1} - p | Z_1, \dots, Z_t)$$

$= W_t$ So $(W_t)_{t \geq 0}$ is a martingale wrt $(Z_t)_{t \geq 0}$

similar for any other indept. $(Z_t)_{t \geq 0}$

$$|W_{t+1} - W_t| = |Z_{t+1} - p| \leq \max\{p, 1-p\} \leq 1$$

By Azuma,

$$P(|W_t - E W_t| \geq x) \leq 2e^{-\frac{x^2}{2t}}$$
$$|X_t - E X_t|$$

Doob martingale construction

let X_t be any rv. and Y_1, Y_2, \dots a random process on Ω

Define $X_n := E(X | Y_1, \dots, Y_n)$ Then $(X_t)_{t \geq 0}$ is a martingale.

To check

$$E(X_{t+1} | Y_1, \dots, Y_t) = E(E(X | Y_1, \dots, Y_{t+1}) | Y_1, \dots, Y_t)$$

wrt $(Y_t)_{t \geq 0}$
 $Y_0 = \emptyset$

Given $Y_1 = y_1, \dots, Y_t = y_t$ (sub-induced prob space) $= E(X | Y_1, \dots, Y_t)$

Restrict to

it's $E(E(X | Y_{t+1})) = EX$ by tower property $= X_t$

Note $X_0 = E(X | Y_0) = E(X)$

A combinatorial setting

Suppose z_1, \dots, z_N are independent RV's in a prob space Ω

Suppose $z_i \in \Omega_i \forall i$

Let $\Pi \Omega_i$ be the prob. space defined by

$$P(z_1, \dots, z_N) = \prod_{i=1}^N P(z_i = z_i)$$

Thm. Suppose X is a rv. on $\Pi \Omega_i$ with

$$X = f(z_1, \dots, z_N) \text{ st.}$$

$$|f(z) - f(z')| \leq c_i \text{ if } z \text{ and } z' \text{ differ on the } i\text{th coordinate}$$

$$\text{Then } P(|X - EX| \geq d) \leq 2e^{-\frac{d^2}{2 \sum c_i^2}}$$

proof: Define Doob martingale

$$X_t = E(X | z_1, \dots, z_t)$$

$$P(X_0 = X_1 \geq \dots)$$

Note $X_0 = EX$

$$X_N = E(X | z_1, \dots, z_N) = X \quad (X = f(z_1, \dots, z_N))$$

check $|X_{t+1} - X_t|$

note

$$\begin{aligned} X_t &= \sum_{z_{t+1}, \dots, z_N} E(X | z_1, \dots, z_t, (z_{t+1}, \dots, z_N = z_{t+1}, \dots, z_N)) P(z_{t+1}, \dots, z_N) \\ &= \sum_{z_{t+1}, \dots, z_N} E(X | z_1, \dots, z_t, z_{t+1}, \dots, z_N) \prod_{i=t+2}^N P(z_i = z_i) P(z_{t+1} = z_{t+1}) \end{aligned}$$

$$X_{t+1} |_{z_{t+1}=z} = \sum_{z_{t+2}, \dots, z_N} E(X | z_1, \dots, z_t, z_{t+1}=z, (z_{t+2}, \dots, z_N = z_{t+2}, \dots, z_N)) \prod_{i=t+3}^N P(z_i = z_i)$$

restricted prob. space

Difference betw. corresponding terms is $\leq c_{t+1}$

So restrict to $z_{t+1} = z$

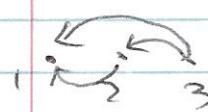
$$|X_t - X_{t+1}| \leq c_{t+1}$$

$$\begin{matrix} X & \text{EX} \\ \downarrow & \swarrow \end{matrix}$$

Assume $\Rightarrow P(|X_N - X_0| \geq \alpha) \leq 2e^{-\alpha^2 / 2ZG^2}$

cor. Let S_i be the edges of K_n from vertex i to $\{1, \dots, i-1\}$

suppose X is a rv. on $G(n, p)$ st.



if $G, H \in K_n$ diff. only on edges S_i ,

then $|X(G) - X(H)| \leq c_i$

then $P(|X - EX| \geq \alpha) \leq 2e^{-\alpha^2 / 2Zc_i^2}$

Proof: Z_i gives the outcome of the coin flips for edges in S_i , in $G(n, p)$

comb. setting thm applies.

June 12 ~~th~~ 2008

Corl. $S_i = \{ik : 1 \leq k \leq i\}$ implies
 Zf $G, H \in K_n$, differing only on S_i always satisfy

$|X(G) - X(H)| \leq c_i$

then $P(|X - EX| \geq \alpha) \leq 2e^{-\alpha^2 / 2Zc_i^2}$
 in $G(n, p)$

(or (shamir & Spencer '87))

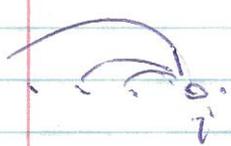
For $G \in G(n, p)$ $P(|X(G) - EX(G)| \geq \alpha) \leq 2e^{-\alpha^2 / 2n}$

(e.g. $\alpha = w(n) / \sqrt{n}$, for any $w \rightarrow \infty$, then Prob $\rightarrow 0$)

X is "concentrated" in interval of width $w\sqrt{n}$

$\frac{n}{2 \log_2 n} (1 + o(1)) \leq X \leq \frac{n}{\log_2 n} (1 + o(1))$

Proof: Zf G, H as in cor 1. k -colouring of G



so $X(H) \leq X(G) + 1$ change S_i according to H &

vire versa. Give i a new colour $\rightarrow (k+1)$ -colouring of H

so $|X(G) - X(H)| \leq 1$

Thm: (Bollobas '88)

Zf $G \in G(n, \frac{1}{2})$, eas $X(G) \sim \frac{n}{2 \log_2 n}$

oas $f \sim g \Leftrightarrow \text{i.e. } \forall \epsilon > 0 \exists \frac{f-g}{g} < \epsilon$, eas.)

$\mathbb{E}(\# k\text{-colourings})$

Let k_0 be the least integer st.

$$\binom{n}{k_0} 2^{-\binom{k_0}{2}} < 1$$

$\mathbb{E} \#$ indep sets of size k_0 we saw $k_0 \leq 2 \log_2 n$ a.s.

For $k=0$ (∞) $k \rightarrow \infty$

$$\binom{n}{k} 2^{-\binom{k}{2}} \sim \frac{n^k}{k! 2^{k(k-1)/2}} \sim \left(\frac{ne}{k 2^{(k-1)/2}} \right)^k \frac{1}{\sqrt{2\pi k}}$$

For $k \leq 2 \log_2 n \rightarrow 2 \log_2(100 \log_2 n) + 1$

$$2^{(k-1)/2} < 2^{\log_2 n - \log_2(100 \log_2 n)} = \frac{n}{100 \log_2 n}$$

$$\text{so } \binom{n}{k} > \frac{e^{100 \log_2 n}}{k} > (25e)^k$$

so $k_0 \sim 2 \log_2 n$

Let $k_1 = k_0 - 4 \sim 2 \log_2 n$

procedure: choose an indep set of size k_1

colour with one colour, delete it

repeat (using new colour for each new set)

when none exists, colour remaining vertices with new distinct k colours

claim: In $\mathcal{G}(n, \frac{1}{2})$ a.s. the number of "remaining vertices" is

at most $\frac{n}{\log_2 n}$

$$\text{In that case, } \chi(G) \leq \frac{n}{k_1} + \frac{n}{\log_2 n}$$

$$\sim \frac{n}{2 \log_2 n}$$

Lemma For $G \in \mathcal{G}(n, p)^{1/2}$ a.s. every set of $\geq \frac{n}{\log_2 n}$ vertices has an indep set of size of k_1 .

This implies the claim.

Proof of lemma: consider k_1 -cliques, (equiv. problem)

on prob.

Plan find bound B that a given set of $T = \lceil \frac{n}{\log^2 n} \rceil$ vertices has no k_1 -cliques, then by union bound,

$$P(\text{no such set exists}) \leq \binom{n}{T} B$$

$$\text{Use } \binom{n}{T} \leq 2^n, \text{ and find } B = e^{-n^{2-O(1)}}$$

Then prob $\rightarrow 0$

Only need to find such B

ie. bound prob (no k_1 -clique) in $G(T, 1/2)$

$\mathbb{E} \# k_1$ -cliques? in $G(n, 1/2)$

$$\binom{n}{k_1} 2^{-\binom{k_1}{2}} = f(k_1) \quad k_1 = k_0 - c \text{ - constant} \quad 2^{c k_0 + O(1)}$$

$$\frac{f(k_0 - c)}{f(k_0)} \sim \left(\frac{k_0}{n}\right)^c \cdot \frac{2^{-\binom{k_0 - c}{2}}}{2^{-\binom{k_0}{2}}} = \left(\frac{k_0}{n}\right)^c \cdot 2^{(k_0 - 1) + \dots - (k_0 - c)}$$

$$= \left(\frac{k_0}{n}\right)^c n^{2c} (1 + o(1)) = n^{c T O(1)} \quad (k_0 = n^{o(1)})$$

$$f(k_0) < 1 \quad f(k_0 - 1) > 1$$

$$\text{So } f(k_0 - 3) \geq n^{3 + o(1)}$$

Lemma 2: $Z_n \subset G(n, 1/2)$, $P(\text{no cliques of size } k_1) < e^{-n^{2-O(1)}}$

$Y = \#$ of k_1 -cliques?

close to $\mathbb{E} Y$?

or in martingale is large



Not Good

Nice trick let $Z =$ cardinality of the largest family of edge-disjoint k_1 -cliques in $G(n, 1/2)$

Lemma 3: $\mathbb{E} Z = n^{2-O(1)}$ (proof later)

$$\binom{n}{2} \sim n^{2-O(1)}$$

Changing one edge changes Z by 1

So using comb. setting

$\Delta_i = \{0, 1\}$ (1 denotes i th edge of $G(n, 1/2)$ present)

$i = 1, \dots, \binom{n}{2}$

$$G(n, 1/2) \leftrightarrow \prod \Delta_i$$

$c_i = 1$ for all i if f is Z

$$\text{So } P(|Z - EZ| \geq tEZ) \leq e^{-\frac{(tEZ)^2}{2\binom{n}{2}}} = 2e^{-\frac{t^2 EZ^2}{n}}$$

$$\text{So } P(Z=0) \leq e^{-n^2 \cdot 0(1)}$$

- gives Lemma 2.

Now define k_1 for $T = \frac{n}{2 \log_2 n}$

$$k_0 = \text{least } k \text{ st } \binom{T}{k_0} 2^{-\binom{k_0}{2}} < 1$$

$$k_1 = k_0 - 4$$

$k_1 \sim 2T \log_2 T$ & by lemma 2,

$P(\text{no cliques of size } k_1)$

June 17th 2008

$$f(k) = \binom{n}{k} 2^{-\binom{k}{2}} = E(\# k\text{-cliques in } G(n, p))$$

$$k_0 = \min \{k : f(k) < 1\}$$

$$k_1 = k_0 - 4 \sim k_0 \sim 2 \log_2 n$$

Lemma 0: $f(k_1) \geq n^{3+o(1)}$

Lemma 1: let $Z = \max$ no of family of edge-disjoint k_1 -cliques in $G(n, 1/2)$
 $EZ \geq n^{2+o(1)}$ (proof soon)

Lemma 2: $P(G(n, 1/2) \text{ has no } k_1\text{-clique}) \leq e^{-n}$

Thm: $\chi(G(n, 1/2)) \sim \frac{n}{2 \log_2 n}$ a.a.s.

proof: Let $T = \frac{n}{2 \log_2 n}$, $k_1 = k_1(T)$

$$k_1 \sim 2 \log_2 T \sim 2 \log_2 n$$

Remove i -sets of size k_1 greedily,

till none remain, one colour each

& colour remaining vertices.

$$T = \frac{n}{\log_2 n} = n^{1-o(1)}$$



31.

P (a set of T vertices with no k_1 -set)
 union bound $\leq e^{-T^2 \cdot o(1)} \binom{n}{T} \leq e^{-n^2 \cdot o(1)} \cdot 2^n \rightarrow 0$

So # remaining vertices $< T$ a.a.s.

$$\# \text{ colours} \leq \frac{n}{k_1} + T \leq \frac{n}{2 \log_2 n} (1+o(1))$$



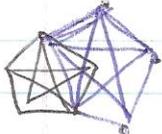
$$\# \leq \frac{\binom{n}{2}}{\binom{k_1}{2}}$$

Proof of Lemma 1:

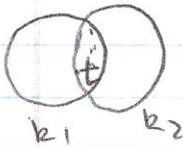
$Z = \max$ size of family of edge-disjoint k_1 -cliques.

$$E(\# \text{ } k_1\text{-cliques}) = f(k_1) \geq n^{3+o(1)}$$

Let $\Delta = E$ # ordered pairs of k_1 -cliques with ≥ 2 vertices in common.



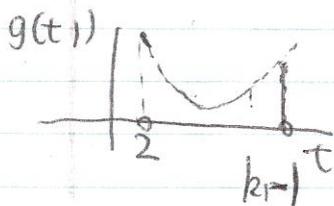
$$\Delta = \sum_{t=2}^{k_1-1} \binom{n}{k_1} \binom{k_1}{t} \binom{n-k_1}{k_1-t} \left(\frac{1}{2}\right)^{2\binom{k_1}{2} - \binom{t}{2}}$$



$$g(t) \sim \binom{n}{k_1} \frac{k_1^2}{2} \frac{k_1^2}{n^2} \frac{1}{2} \geq \binom{k_1}{2} - \binom{t}{2}$$

$$\frac{\binom{n}{k_1-1}}{\binom{n}{k_1}} = f(k_1) \cdot \frac{k_1^4}{n^2} \geq n^{3+o(1)} \cdot f(k_1) > f(k_1) n^{1+o(1)}$$

$$g(3) \sim f(k_1)^2 \frac{k_1^3}{6} \cdot \frac{k_1^3}{n^3} \times 8 = o(g(2))$$



$$g(k_1-1) \sim \binom{n}{k_1} k_1 n \cdot \left(\frac{1}{2}\right)^{\binom{k_1}{2} + \binom{k_1-1}{2}}$$

$$= f(k_1) k_1 \cdot n \cdot \left(\frac{1}{2}\right)^{2 \log_2 n (1+o(1))}$$

$$< f(k_1) (1+o(1)) \left(\frac{1}{n}\right)^{2+o(1)}$$

$$= o(g(2))$$

$$\text{Let } k_t = \frac{g(t+1)}{g(t)}$$

$$g(t) = \frac{\text{const} \cdot 2^{\binom{t}{2}}}{t! (k_1 - t)!^2 (n - 2k_1 + t)!}$$

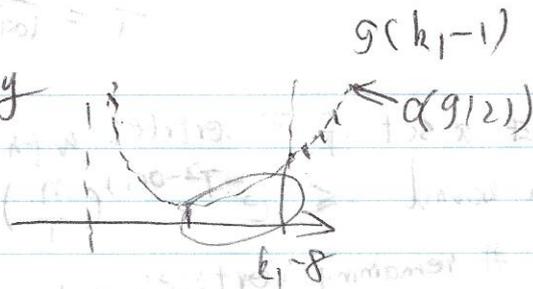
$$= \frac{2^t (k_1 - t)^2}{(t+1)(n - k_1 + t+1)} \leftarrow \Theta(\log^2 n)$$

Certainly $< \frac{1}{2}$ for $t < \frac{\log_2 n}{2}$

Hilroy

for $t \geq \frac{\log_2 n}{2}$ till $t = k_1 - 8$ say

$\frac{R_{t+1}}{R_t} > \frac{3}{2}$ so $g(t)$ is convex



For $t = k_1 + \text{const} = 2 \log_2 n + \text{const}$

$$R_t = n^{1+o(1)}$$

$$\text{So } \Delta \sim g(2) = f(k_1) \frac{k_1^4}{n^2}$$

$E \# k_1\text{-cliques} = f(k_1)$

E ordered pairs intersecting $\sim \Delta$

create random family H of k_1 -cliques.

— choose each k_1 -clique with prob. q

$$E |H| = f(k_1) q$$

$$E \# \text{ unordered pairs in } H \sim \frac{\Delta}{2} \cdot q^2$$

edge-intersecting

delete one from each, to get H' .

H' : edge-disjoint

$$E(H') = E(H) - E(\# \text{ deleted})$$

$$= f(k_1) q - \frac{\Delta}{2} q^2$$

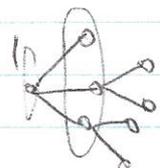
$$\frac{f(k_1)}{\Delta} \sim \frac{n^2}{k_1^4 f(k_1)}$$

$$\text{set } q = \frac{f(k_1)}{\Delta} \rightarrow 0$$

$$E(H') = \frac{f(k_1)^2}{2\Delta} = \frac{n^2}{2k_1^4} = n^{2+o(1)}$$

$G(n, p)$

$p \sim \frac{c}{n}$
Galton-watson
branching process



Start with 1 individual. At each step, all individuals

give birth & die.

All birthings indept

How many children distributed as RV. Z .

$Z \geq 0$ always, with $E Z < \infty$

Thm. $P_Z = \text{prob}(\text{extinction})$

Let $X_t = \text{no of individuals after } t \text{ steps}$

Then $P_z = P(\exists i \geq 0 : X_i = 0)$

Thm. (i) $Ez < 1$, then $P_z = 1$

(ii) $Ez = 1$ then $P_z = 1$ unless $P(Z > 0) = 1$

(iii) $Ez > 1$ then $0 < P_z < 1$ (unless $P(Z > 0) = 1$)

proof of (i) $X_0 = 1$
 $E X_1 = Ez$

$\& P_z = 0$

$$E X_2 = \sum_{i \geq 0} P(X_1 = i) (i Ez)$$

$$= Ez E X_1 = (Ez)^2$$

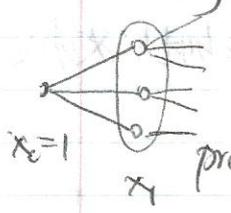
induction $\rightarrow E X_t = (Ez)^t \rightarrow 0$

$P(X_t > 0) \rightarrow 0$ $P_z > 1 - \epsilon$ $\forall \epsilon > 0$

So $P_z = 1$

June 19th, 2007

Branching processes:



Z : dist of # of children

$P_z = P(\exists t > 0 : X_t = 0)$

proof of thm(iii) ($P(Z=0) > 0, Ez > 1$)

say an individual "fails" if it has finitely many descendants.

$P_z = P(\text{first individual fails}) = P(\text{all } X_1 \text{ children fail})$

$$= \sum_{i \geq 0} P(X_1 = i) (P_z)^i$$

pgf for $Z = \sum_{i \geq 0} P(Z=i) x^i =: f_Z(x)$ $f_Z(1) = 1$

So $P_z = f_Z(P_z)$ So $P_z = 1$

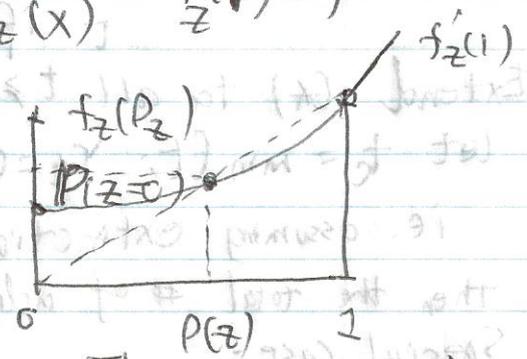
Note $f_Z'(x) = \sum_{i \geq 1} P(Z=i) i x^{i-1}$

$$f_Z'(1) = E(Z) > 1$$

f' is inc for $x > 0$

$\Rightarrow f$ is convex

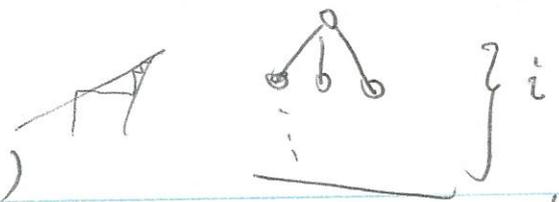
f is inc too



The soln. (Unique)

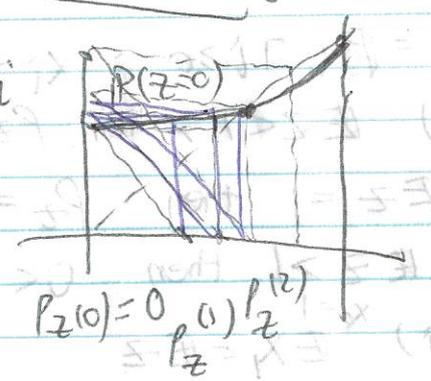
$0 < P_z < 1$

let $P_z^{(i)} = P(X_{A1} = 0)$

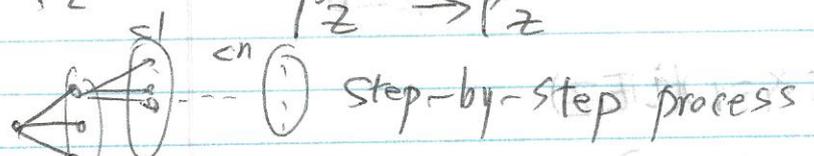


$$P_z^{(2)} = \sum_{i \geq 0} P(Z=i) P_z^{(1-i)}, i$$

$$= 1 + P_z^{(1-1)}$$



$P_z^{(0)} = 0$
 $P_z^{(i)} \rightarrow P_z$

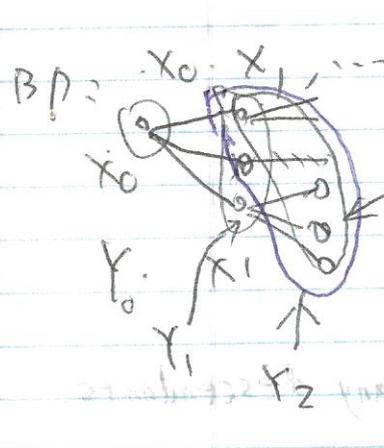


Step-by-step process

Let $Y_0 = 1$

$$Y_t = Y_{t-1} + Z_t - 1 \quad (t \geq 1)$$

(Z_t & Z all Z_t 's indept.)



BP: X_0, X_1, \dots $Y_i = \#$ of "live" individuals
 where at each step, pick an individual
 Y_3 is "current" generation, let it give birth & die
 Note $X_0 = Y_0 = 1$

$$X_1 = Y_1 = X_{Y_0}$$

$$X_2 = Y_2 = X_{Y_1}$$

$$X_t = Y_{X_{t-1} + \dots + X_{t-1}}$$

Note $Y_t = Z_1 + Z_2 + \dots + Z_t - (t-1)$ (*)

[if $Y_1, \dots, Y_{t-1} \geq 1$]

Extend (*) to all $t \geq 0$ regardless of $Y_i \leq 0$

let $t_c = \min(t; Y_t = 0)$ (if it exists)

i.e. assuming extinction

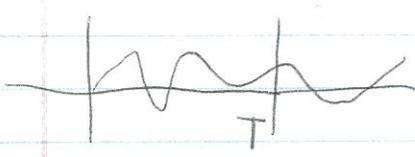
then the total # of individuals in BP is $|t-1| + 1 = t$

Special case:

$$Z \sim P_0(s)$$

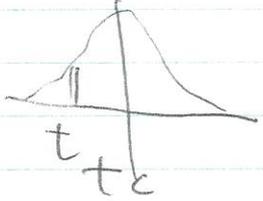
lemma let $c > 1$, $\forall \epsilon > 0 \exists N(\epsilon) : \text{if } BP \text{ reac}$
 $IP(BP \text{ reaches contains in total at least } N(\epsilon) \text{ individuals}$
 $\& \text{ subsequently goes extinct}) < \epsilon$

event $E_{N(\epsilon)}$
 proof = E implies $\{ \exists t > T, Y_t \leq 0 \}$



$$P(Y_t \leq 0) = P\left(\sum_{i=1}^t z_i \leq t-1\right)$$

$$\sum_{i=1}^t z_i \stackrel{d}{\sim} P_0(ct)$$



$$\leq t P(P_0(ct) = t)$$

$$= t e^{-tc} \frac{(tc)^t}{t!} \ll O\left(t \frac{(tc)^t e^{-tc}}{\left(\frac{t}{e}\right)^t}\right)$$

$$= O(t) (ce^{-c})^t$$

Now $\frac{c}{e^c} < e^{-c} \Rightarrow 1 + (c-1) + \frac{(c-1)^2}{2} < e^c$

$$\leq \frac{c}{1 + (c-1) + \frac{(c-1)^2}{2}} =: c' < 1$$

This is $< \epsilon$ for t suff. large i.e. $\gg N(\epsilon)$

Similarly $\sum_{t \geq T} O(t) (ce^{-c})^t < \epsilon$ for T suff. large.

so by union bound $IP(\exists N(\epsilon), t \geq N(\epsilon), Y_t < 0) < \epsilon$

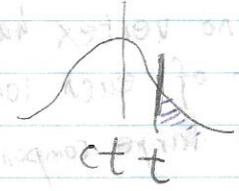
$$IP(\exists t \geq N(\epsilon) : Y_t < 0) < \epsilon$$

Lemma 2: if $c < 1$

$$\exists c^* < 1 : \exists A(c^*) > 0 \forall t$$

$$A(\text{constant}) P(Y_t \geq c)$$

proof: $IP(Y_t > c) = IP(P_0(ct) \geq t)$



Milroy

$$= \sum_{i \geq t} e^{-tc} \frac{(tc)^i}{i!}$$

$$\leq \frac{1}{1-c} e^{-tc} \frac{(tc)^t}{t!} = O((ce^{-c})^t)$$

put $1-c = \varepsilon$
 $(\varepsilon > 0)$

$$ce^{-c} = (1-\varepsilon)e^\varepsilon = \exp(\log(1-\varepsilon) + \varepsilon)$$

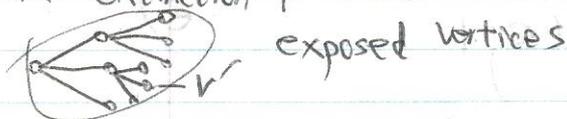
$$\leq \exp\left[-\varepsilon - \frac{1}{2}\varepsilon^2\right] < 1$$

June 24th 2008. Random Graph G0739 Nick Wormald.

Giant Components etc., largest Component?

$$p = \frac{c}{n} \quad G(n, p)$$

BP, extinction prob P_z z "birth rate"



Graph search process: start vertex v , find neighbours, pick a neighbour & look for its, etc (BFS)

If k vertices exposed, when $N(v)$ are investigated, distribution is

$$\text{Bin}(n-k, p) \approx \text{Bin}(n, p) \text{ if } k, p \frac{c}{n} \text{ is small.}$$

Lemma: $\text{Bin}(n, \frac{c}{n}) \xrightarrow{d} P_0(c)$ if c fixed ($n \rightarrow \infty$)

Proof: $\binom{n}{k} \left(\frac{c}{n}\right)^k \left(1 - \frac{c}{n}\right)^{n-k} \sim \frac{n^k}{k!} \frac{c^k}{n^k} e^{-c}$

what we expect:

1. $c < 1$ Consider BP with $z \sim P_0(c)$

$$E X_t = (E Z)^t = c^t \quad \text{so } P(X_t \geq 1) \leq c^t \text{ (Markov)}$$

\uparrow
t-th generation

eg. $t \approx \log_c \left(\frac{1}{n^2}\right)$ prob $\approx \frac{1}{n^2}$

"so" aas. no vertex has a BP this long.

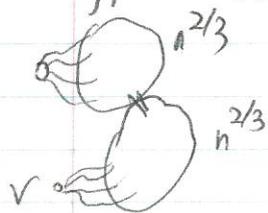
so diameter of each component $\leq C \log n$ aas.

~~not~~ permits large components

- use step by step process.

$Y_0 = 1, Y_t = Y_{t-1} + Z_{t-1}$ = no. of vertices still to explore
 after processing t . \uparrow Po(cc)

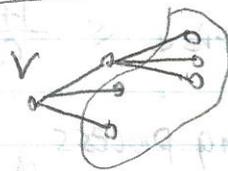
$P(Y_t > 0) \rightarrow 0$ exp. fast. (last proof)
 so. $\exists \#$ vertices with $> C \log n$ component size is $O(1)$
 2. $c > 1$ step-by-step Y_t , if Ct reaches much ^{little} 0 ,
 bigger than $C \log n$, then $P(\text{ever reaches } 0) = O(\frac{1}{n})$ (last proof)
 - suggest each component either $O(\log n)$ or is the 'giant'



$\Pr(v \text{ is in a small component}) \approx P_Z$ for $Z \sim \text{Po}(c)$
 BP not exact

Case 1: $c < 1$.

graph search process



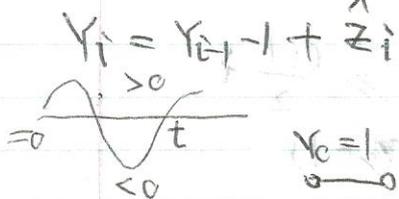
step by step $G_{i+1} = G_i - 1 + Z_i$

$G_i = \#$ uninvestigated vertices after i steps.

$(G_0 = 1 \text{ etc. } G_i = G_{i-1} - 1 + Z_i)$

$P(G_i \leq 0) \geq P(Y_i \leq 0)$

step by step with $\hat{Z}_i \sim \text{Bin}(n, p)$

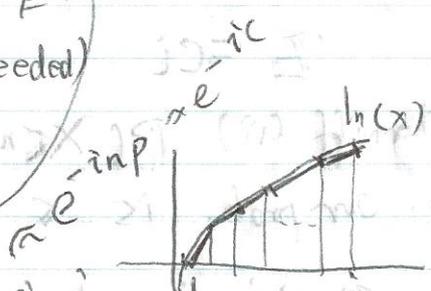


$P(Y \leq 0) = P(\sum_{j=1}^i \hat{Z}_j \leq i-1)$
 $= P(\text{Bin}(in, p) \leq i-1)$

Lemma: $P(X \geq t) = P(\dots \geq t)$ (upper bound needed)

Let $X \sim \text{Bin}(n, \frac{c}{n})$ Then for $t > c$,

$P(X \geq t) \leq \frac{1}{1 - c/t} (\frac{ec}{t})^t$



$P(\text{Bin}(in, p) \geq t) = \sum_{j \geq t} \binom{in}{j} p^j (1-p)^{in-j}$

$\leq \sum_{j \geq t} \frac{(in)^j}{j!} p^j$ ($j! \geq (\frac{j}{e})^j \cdot e$)

$\leq \sum_{j \geq t} (\frac{ein}{j})^j (\frac{c}{n})^j$

Ratio $1 \rightarrow 11$ $\frac{pin}{j} \leq p1 = c$
 $\Rightarrow \text{sum} \leq \text{1st term} \times \frac{1}{1-c}$

so $\sum \leq \left(\frac{e^{\epsilon c}}{\epsilon}\right)^i = e^{\epsilon c}$

Chernoff (i) $P(X > np + t) \leq e^{-t^2 / 2(np + t/3)}$

proof - like martingale, $t = i(1-p) = i(c)$

$P(Bin(i, p) \geq ip + i(1-p)) \leq e^{-i^2(c)^2 / 2(ip + i(c)/3)}$

$= e^{-i \frac{(1-c)^2}{2(c+1/3)}} = e^{-i c_0}$

So $P(v \text{ is in component of } \geq 4) \leq e^{-i c_0}$

So if $i > \frac{1+\epsilon}{c_0} \log n$ prob. = $O(1/n)$

So aas. all components have size $\leq \frac{1+\epsilon}{c_0} \log n$

Case 2: $c > 1$

consider the step by step branching process

Y_0, Y_1, \dots with $Y_t = Y_{t-1} + Z_t$, $Z_t \sim Bin(n, \frac{c}{n})$ ($c > 1$)

Lemma: let $k_0 = \frac{c}{(1+\epsilon)^2} \log n$, $k_1 = \frac{2}{3} k_0$

[expect $Y_i \approx (c-1)i$], then Probability $\geq 1 - O(1/n^2)$

$Y_i \geq \frac{c-1}{2} i$ for $k_0 \leq i \leq k_1$

proof: $P(Y_i < \frac{c-1}{2} i) = P(Z_1 + \dots + Z_i \leq (\frac{c-1}{2} i + i - 1))$

$\mathbb{E} = ci$ $\stackrel{d}{\sim} Bin(i, \frac{c}{n})$

Chernoff (ii) $P(X \leq np - t) \leq e^{-t^2 / 2np}$

so. our prob. is $\leq e^{-((\frac{c-1}{2})^2 / 2ic)} = e^{-((c-1)^2 / 8c) i}$

$\frac{(c+1)i}{2} = ip - t = ci - t \Rightarrow t = \frac{c-1}{2} i$

For $i > \frac{1+\epsilon}{(c-1)^2 / 8c} \log n$ etc, this is $\leq \frac{1}{n^2} e^{-\frac{1}{2} i}$

so Prob is $O(1/n^2)$

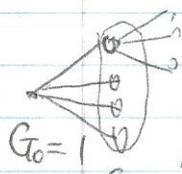
June 26th, 2008 CO739 Graph Theory

Y_t step by step BP $Z \sim \text{Bin}(n, \frac{c}{n})$

Lemma: $P(\exists i \geq c_0 \log n : Y_i = \frac{c-1}{2} i) = O(n^{-2})$

$c_0 = \frac{17c}{(c-1)^2}$

Let $(G_t)_{t \geq 0}$ be the graph search process in $G(n, \frac{c}{n})$ $c \geq 1$
(step by step)



$G_t \stackrel{d}{\sim} \text{Bin}(n-1, p)$

$G_{t+1} = G_t + 1 + \text{Bin}(n - [G_t + 1], p)$

Lemma: let $k_0 = 2c_0 \log n$, $k_1 = \lfloor n^{2/3} \rfloor$

Then $P(\exists t, k_0 \leq t \leq k_1 : G_t \leq \frac{c-1}{4} t) = O(n^{-2})$

proof: let $(Y_t^-)_{t \geq 0}$ be the step by step BP with $Z \sim \text{Bin}(n - (c+1)k_1, p)$ ($p = \frac{c}{n}$)

couple Y_t^- with G_t : use the first $n - (c+1)k_1$ "coins"

on edges in the G_t process for Y_t^- — up until $G_t + t \geq (c+1)k_1$ (then stop). or $[G_t \text{ becomes } 0]$

suppose $1 \leq G_t \leq \frac{c-1}{4} t$ with $t \leq k_1$ $t \geq k_0$

Now G_{t+t} never decreases, $G_{t+t} \leq G_t + t$ for $t \leq i$.

$\leq (\frac{c-1}{4})^i < (c+1)i \leq (c+1)k_1$

so coupling works till time i .

so $Y_t^- \leq G_t \leq \frac{c-1}{4} t$ with $t \geq k_0$ this has probability $O(\frac{1}{n^2})$

$Y_t^- : Z \sim \text{Bin}(n', \frac{c}{n})$ $n' = n - (c+1)k_1$ $p = \frac{c}{n} = \frac{c'}{n'}$

$c' = n'p$ $n' \sim n$ $c' \sim c$ $c'_0 = \frac{17c'}{(c'-1)^2} \sim c_0$

lemma implies $P(\exists i \geq c_0 \log n' \text{ with } Y_t^- \leq \frac{c'-1}{2} i) = O(n'^{-2}) = O(n^{-2})$

This proves the lemma. $\left(2c_0 \log n \right) > \frac{c-1}{4} i$ for large n .
 $k_0 \quad k_1 = \lfloor n^{2/3} \rfloor$

con. a.s. all components are small or large.

So pick v for process G_T

$P(v \text{ not in small or large}) = O(n^{-2})$, so $P(\exists v \text{ not in small or large}) = O(n^{-1})$

Small component: size $\leq k_0$

or large) = $O(n^{-1})$

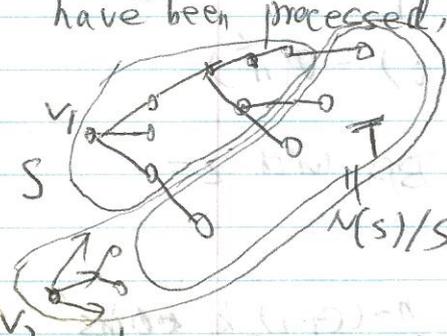
Large component: size $\geq k_1$

Thm: $P(\exists \text{ two})$

A.i.a.s, there is only at most one large component.

proof: Let $v_1, v_2 \in V$

Apply process (G_T) to v_1 , but stop if & when k_1 vertices have been processed.



if don't reach k_1 , terminate with "succeed".

(v_1 not in large component)

if reach k_1 & $|T| \leq \frac{c-1}{4} k_1$, return "fail"

else: (proceed) if $v_2 \in S \cup T$, return "succeed"

else: The random graph restricted to $V \setminus S$ is a

$G(n-k_0, p)$ random graph. indep't of previous steps

Run (G_T) on V_2 . $n' = n - k_0$ $c' = n'p = (n-k_0)\frac{c}{n} \sim c$

large: size $\geq \frac{k_1}{2}$

$k_0' \leq k_1' \leq \frac{k_1}{2}$ in here

Then $P(G_T \text{ large}) \leq \frac{c'-1}{4} (n')^{c'-2} (\frac{k_1}{2}) = O(n'^{-2}) = O(n^{-2})$

return "succeed"

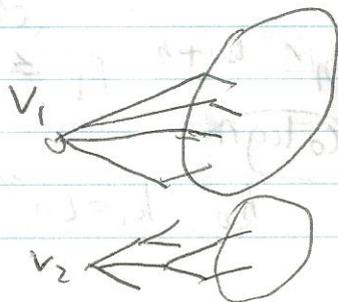
by previous result.

So with prob $(1 - O(n^{-2}))$, if reach $k_1/2$, $|N(s') \setminus S'| \geq \frac{c-1}{8} \cdot \frac{k_1}{2}$

(if don't reach $k_1/2$, v_2 is not in large component) return succeed.

stop at $|S'| = k_1$

if succeed,



$|T| \geq \frac{c-1}{4} k_1$

$|T| \geq \frac{c-1}{16} k_1$

if $T \cap T' \neq \emptyset$, then v_1, v_2 in same component.

Return succeed.

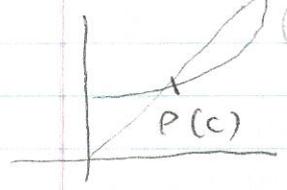
(if NOT) else $P(\text{no edge between } T \& T')$
 $= (1-p) \frac{(c-1)^2}{64} k_1^2 < \exp(-\frac{c}{n} \cdot n^{1/3})$
 $= e^{-\Theta(n^{1/3})} = O(n^{-2}) \rightarrow \text{return fail.}$

otherwise v_1, v_2 in the same component.

How big is largest component?

let $P(c)$ be ext. prob. for (X_t) BP with $Z \sim P_0(c)$

Then $P(c) = f(P(c))$ where $f(x) = \sum_{i \geq 0} \frac{e^{-c} c^i}{i!} x^i$
 $= e^{-c} \sum \frac{(cx)^i}{i!} = e^{cx-c} = e^{c(x-1)}$



let $\hat{P}(c)$ be the P when $Z \sim \text{Bin}(n, p)$

$\hat{P}(p) = \hat{f}(\hat{P}(p))$

$\hat{f} = \sum_{i=0}^n \binom{n}{i} (\frac{c}{n})^i (1-p)^{n-i} x^i = \sum_{i=0}^n \binom{n}{i} (px + (1-p))^n$

for $p = c/n$ $\hat{f}(x) = (1 + p(x-1))^n \sim e^{c(x-1)}$ ($n \rightarrow \infty$)

$\hat{f} \rightarrow f$ pointwise ($x \in (0, 1)$)

so $\hat{P}(\frac{c}{n}) \rightarrow P(c)$

Thm. Let $v \in V(G_{cn, c/n})$. $P(v \text{ is in a small component})$

$\rightarrow P(c)$

proof: if v is in small component, then $G_{K_0} = 0$

so $y_{k_0}^- = 0$ ($Z \sim \text{Bin}(n - (k_0+1)k_0, p)$)

Prob $\leq \hat{P}(\frac{c}{n - (c+1)k_0}) \rightarrow P(c)$

ie. $P(v \text{ is in small component}) \leq P(c) + o(1)$

Recall (Y_t) with $\geq \mathcal{B}_n(n, p)$, then

coupled with G_t , $Y_t \geq G_t$

(Y_t) Extincts with prob. $\sim P(c)$

Prob (this happens after reaching k_0) is $O(n^{-2})$ by lemma.

So $P(\text{extinct by } k_0) \sim P(c)$

So $P(G_t \text{ dies by } k_0) \geq P(c) + o(1)$

↑ ie. v is in small component

Cor. Expected size of ~~giant~~ is ~~$n(P(c))$~~
 $\#$ of vertices in small components
 is $nP(c)$

Thm. a.s. \exists large component with $n(1-P(c) + o(1))$
 vertices.

July 3rd 2008 Random Graph, CO739 / Vick Wormald

$k_0 \approx C \log n$, $k_1 \approx c, n^{2/3} > \frac{1}{2}$

$G(n, \frac{c}{n}), c > 1$

AAS. all component sizes between k_0 & k_1 are "small"

$\mathbb{E} \#$ vertices in small components $\sim P_2 \cdot n \stackrel{d}{\sim} P_0(c)$

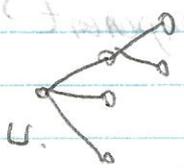
$\mathbb{E} \#$ vertices in giant $\sim (1 - P_2) \cdot n$

AAS. unique giant.

Let X be no. of vertices in components of size $\leq k_0$

Have $\mathbb{E} X \sim P_2 \cdot n$ (very small)

$\mathbb{E} X(X-1) = \sum_{\substack{u, v \in V \\ u \neq v}} P(u, v \text{ in small components})$



Graph exploration. But stop if reach $> k_0$ ($k_0 + 1$) vertices from u .

$P(\text{stop}) \sim P_2$ (last time)

$P(v \text{'s component fully found}) \sim P_0$

if step with a very small components C .

$\leq k_0 + 1$
u

$P(v \text{ in } C) = O(\frac{k_0}{n}) = o(1)$

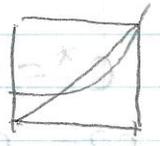
otherwise, apply exploration from V .

$P(v \text{ is in small component}) \sim P_z$ by earlier results

$\sim n^{-O(\log n)}$

$G(n', \frac{c}{n})$ $n' = n - |C| = n - O(\log n)$

$P_0(c')$



$= G(n', \frac{c'}{n'})$ where $\frac{c'}{n'} = \frac{c}{n}$ $c' = \frac{cn'}{n} \sim c$

so this is $\sim P_z$ for $z \sim P_0(c)$

Thus $E X(X-1) \sim n^2 P_z^2$ Note $E X \rightarrow \infty$
 $\sim (E X)^2$

$\text{var } X = E X^2 - (E X)^2 = o((E X)^2)$

$\forall \epsilon > 0$ a.s. $|X - E X| \leq \epsilon E X$ by Chebyshev.

$P(|X - E X| \geq t) \leq \frac{\sigma^2}{t^2}$

$\exists g(n) \rightarrow 0$ a.s. $|X - E X| \leq g(n) E X$ exercise.

"a.s. $X \sim E X$ " (defn)

so a.s. $P_z \cdot n$ vertices are in V small components & $(1 - P_z)n$ in a unique "giant" component.

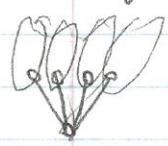
$c > 1$ $G(n, \frac{c}{n})$



Consider "pure" BP with $z \sim P_0(c)$ (conditioned on extinction)

$z \sim P_0(c)$ An individual "fails" if it has a finite # of descendants

$P(\text{fail}) = P_z$ by independence



$P(\text{offspring which all fail}) = \frac{e^{-c} c^i}{i!} P_z^i$

$= \frac{e^{-c} (c P_z)^i}{i!} \dots (A)$

$P(\text{all fail}) = \frac{e^{-c} \sum_i \frac{(c P_z)^i}{i!}}{e^{-c} \sum_i \frac{c^i}{i!}} = e^{-c} \cdot e^{c P_z} = e^{c(P_z - 1)}$ Hilroy (B)

so condition on all failing

$$P(i \text{ children}) = \frac{A}{B} = \frac{e^{-cP_2} (cP_2)^i}{i!}$$

so this is a BP with $\hat{z} \sim P_0(cP_2)$

Recall pgf for z is $e^{c(x-1)}$

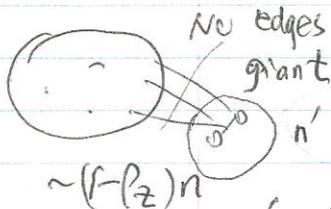
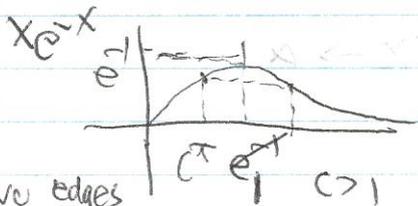
so $P_2 (=P)$ satisfies

$$P = e^{c(P-1)}$$

Defn. $c^* = cP_2$ ($\hat{z} \sim P_0(c)$) is the conjugate of c ($c > 1$)

$$\text{so } cP = c e^{c(P-1)}$$

$$cP e^{-cP} = c e^{-c} \quad c^* e^{-c^*} = c e^{-c}$$



looks like $G(n', c/n) = G(n', c'/n')$
 where $\frac{c'}{n'} = \frac{c}{n} = P$ $c' = \frac{c n'}{n}$
 $c \approx c^*$ $n' \sim P_2 n$ $c' \approx c P_2$ ✓

Not quite rigorous above.

Let Q be a property depending only on isomorphism class of the graph (not on graph labels)

Thm. Assume that $n' = n'(n)$ ($n' \sim nP$) implies $G(n', \frac{c}{n})$

$c > 1$ fixed, and

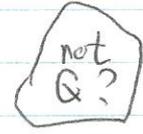
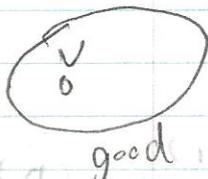
Then a.s. $G(n, \frac{c}{n}) \setminus C$ has Q , where C is the giant.

proof: $\exists g(n) \rightarrow 0$ st. a.s. unique giant exists, of size

$$n(1 - g(n))$$

call such a component "good"

let $v \in V$, let C_v be v 's component



$P(C_v \text{ is good} \mid G - C_v \text{ does not have } Q)$

$$= \sum_{\text{good contain } v} P(\hat{C} = C_v) \times P(G - \hat{C} \text{ does not have } Q)$$

$\mathbb{P}(G \pm g(n))$ $\mathbb{P}(G - v(C))$ does not have Q
 $p = \frac{c}{n}$ $\mathbb{P}(G(n - |C|, \frac{c}{n})$ does not have Q
 let \hat{n} be the value of n maximizing the second prob.

Then $\hat{n} \sim np$ so the prob $\rightarrow 0$

$$= \sum_{C \text{ good comp}} \mathbb{P}(C^{\wedge} = C) \cdot o(1) = o(1)$$

so $\mathbb{E}(\# \text{ vertices in good components with rest of graph not sat. } Q)$
 $\mathbb{E} = o(n)$ (little)

let T be $\mathbb{P}(\exists \text{ \textit{good} } C, \text{ and } G - C \text{ does not have } Q)$

Given T , # vertices as above is $\sim (1-p)n$

But $E \geq T \times \mathbb{E}(\text{vertices as above} \mid S)$
 $o(n) \geq o(n) \geq cn$

Corollaries

vertices of degree k in giant is a.s. $C_k n (1 + o(1))$

$\mathbb{P}(\# \text{ cycles outside giant} > wn) \rightarrow 0$

for all functions $w \rightarrow \infty$

$$\mathbb{E}(\# k\text{-cycles in } G(n, \frac{c^*}{n})) \sim \frac{(c^*)^k}{2k}$$

look at # in $G(n, p)$
 - # outside giant
 (subtract)
 $G(n', \frac{c^*}{n'}) \sim G(n, \frac{c^*}{n})$
 $p \sim \frac{c^*}{n}$

$$\sum_{k \geq 3} \frac{(c^*)^k}{2k} = o(1)$$

$$\sum_{i \geq 1} \frac{(c^*)^i}{i} = -\log(1 - c^*)$$

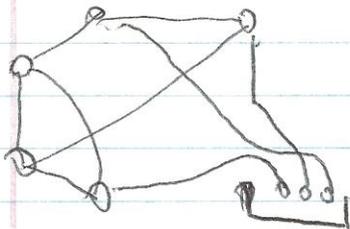
$G(n, \frac{c}{n})$ $c > 1$
 $\geq n/2$ let in the giant
 very few outside

July 8th 2008 Random Graph (0739)

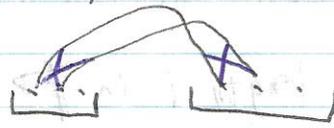
$$= \frac{(nd)!}{\left(\frac{nd}{2}\right)! 2^{nd/2}} \quad \text{Only consider } nd \text{ even!} \quad \cdot 47$$

Given a simple graph G ,

$$|\{P: G(P) = G\}| = (d!)^n$$



$d!$ ways to "wire up" a vertex.



so in $\mathcal{P}_{n,d}$, $P(G(P) = G) = \frac{(d!)^n}{(nd-1)!!}$

Cor. If H is a property of simple graphs $\subseteq \mathcal{G}_{n,d}$



then $P_G(H) = \frac{P_P(H)}{P(\text{simple})}$

Let $H = \{G\}$ for a particular graph G .

$$P_G(H) = \frac{1}{|\mathcal{G}_{n,d}|} = \frac{(d!)^n}{(nd-1)!!} \cdot \frac{1}{P(\text{simple})}$$

$$\text{so } |\mathcal{G}_{n,d}| = \frac{(nd-1)!!}{(d!)^n} \cdot P(\text{simple}) = \frac{(nd)!}{\left(\frac{nd}{2}\right)! 2^{nd/2} \cdot (d!)^n} \cdot P(\text{simple})$$

Lemma: If S is a set of s disjoint pairs of points, then

$$P_P(S \subseteq P) \text{ is } \frac{1}{(nd-1)(nd-3)\dots(nd-2s+1)}$$

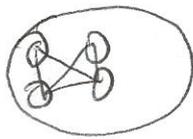
where $s = |S|$
 proof: # P containing S is $(nd-2s-1)!! \cdot (nd-1)!!$

Note if $s = o(\sqrt{n})$, $P(S \subseteq P) \sim (nd)^{-s}$

Asymptotics $n \rightarrow \infty$, d fixed.

Lemma: If G_0 is a graph with v vertices and μ edges, then

$$P_P(G(P) \supseteq G_0) = O(n^{-\mu}) = o(1) \quad \text{Hilroy}$$



Proof: Expected no. of copies of G_0 in $G(P)$ is $\frac{1}{n \cdot d!}$

$$\leq n^{\underbrace{d}_{\text{bounded}}} \cdot \underbrace{(nd)^{-d}}_{\substack{\text{IP (pairs in } P)} \\ \# \text{ ways to choose pairs giving } G_0}}$$

$\#$ ways to choose pairs giving G_0



Let X_i = no. of "cycles of length i in P ."

"Cycle" = set of pairs giving a cycle in $G(P)$

length 1: loop

length 2: double edge

$Z_n P_{n,d}$

$$E X_1 = \frac{n \binom{d}{2}}{nd-1}$$

by lemma $\sim \frac{d-1}{2}$

$$E X_2 = \binom{n}{2} \binom{d}{2}^2 \cdot 2 \times \frac{1}{(nd-1)(nd-3)} \sim \frac{(d-1)^2}{4}$$



2 ways

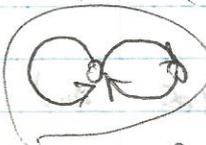
$$E X_i = \binom{n}{i} i^i \cdot \underbrace{d(d-1) \dots d(d-1)}_{[d(d-1)]^i} \cdot \frac{1}{(nd-1)(nd-2i+1)} \cdot \frac{1}{i!}$$

specify one point in the basket

(in 2^i ways) $\sim \frac{(d-1)^i}{2^i}$

i factors $\sim (nd)^i$

$$E X_1 X_2 \sim (E X_1) (E X_2)$$



$E \neq 0$ terms with intersecting vertices contribute

$o(1)$ by lemma

method of moments $\Rightarrow X_1, \dots, X_k$ are asymptotically indept. 49.

Poisson with means $\lambda_1, \lambda_2, \dots, \lambda_k$.

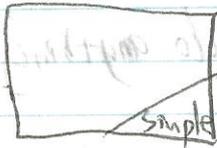
$$\lambda_i = \frac{(d-1)^i}{2^i} \quad \text{(tree like...)} \quad \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array}$$

$$\Rightarrow P(\text{simple}) = P(X_1 = X_2 = 0) \sim e^{-\lambda_1 - \lambda_2} = e^{-(d^2-1)/4}$$

Cor. If H is a property of ~~simple~~ ^{multi} graphs, and $P \in \mathcal{P}_{n,d}$ has $G(P) \in H$ a.s., then

$G \in \mathcal{G}_{n,d} \in H$ a.s.

$\mathcal{P}_{n,d}$



$$P(G \in H) = P_P(G(P) \in H \wedge G(P) \text{ simple})$$

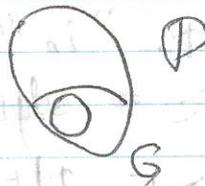
$P_P(\text{simple})$

60739 Random Graph Prof. Nick Wormald July 10th 2008 sunny

$\mathcal{P}_{n,d}$ $\mathcal{G}_{n,d}$ \mathcal{P}

n buckets

$G(\mathcal{P}_{n,d})$



$$P(\text{simple}) \sim e^{-(d^2-1)/4}$$

Connectivity

no edges



J
 $|J|=j$



S
 $|S|=s$

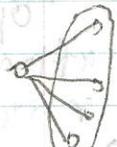


A (J, S) -separator

$$J \subseteq V \quad S \subseteq V \setminus J$$

No edge from J to $V - (S \cup J)$

$|J|=j \quad |S|=s$ all $v \in S$ are adj. to $\geq 1 v \in J$



d -regular

Independence property of $\mathcal{P}_{n,d}$



$(1, d)$ -separator

trivial

repeat:

a pairing

consider any algorithm generating

$P \in \mathcal{P}_{n,d}$ as follows.

Step 1. choose an unmatched point using any rule.
(can depend on previous pairs chosen)

step 2: choose a mate for a ^(uniform) UAR x from remaining pts. until all points matched

Then the pairing is dist. as $P_{n,d}$

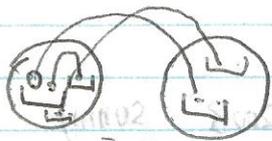
Let $X_{j,s}$ be the number of (j,s) -separators in $P_{n,d}$

Then $E X_{j,s} = \sum_{\substack{J \cup S \\ |J|=j, |S|=s}} P(Y_{J,S} = 1)$
 indicator for J, S being separator.

Fix $d \geq 3, s \leq d-1$

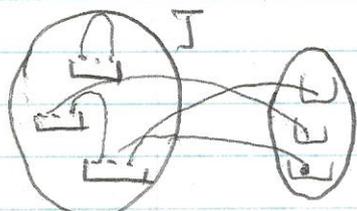
Fix J, S , use the algorithm for generating P , with rule

Choose a point in a bucket in J . After, do anything



For J, S to be separator each mate chosen must be "in" $J \cup S$.

Until all points "in" J are matched.



Note # edges inc. with ≥ 1 vertex in J

$$\geq \lceil \frac{j+d+s}{2} \rceil = r$$

$j \cdot d$

$\geq s$

$J \cup S$ "contain" $(j+s)d$ pts.

P has nd points

$$P(J, S \text{ separate}) \leq \prod_{i=1}^r \frac{(j+s)d - (2i-1)}{nd - (2i-1)} \quad (= \text{Prob OK after } r \text{ steps})$$

$$\leq \prod_{i=1}^r \frac{(j+s) - \frac{2i-1}{d}}{n - \frac{2i-1}{d}}$$

d : even

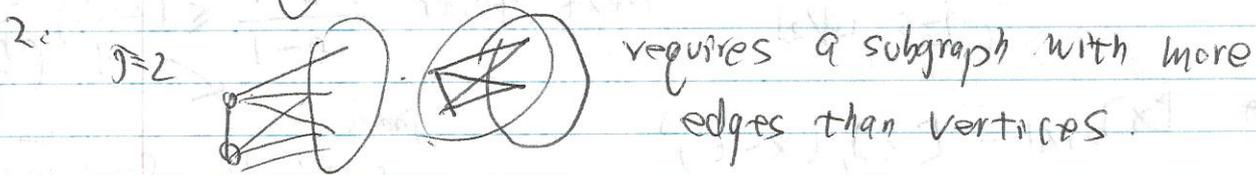
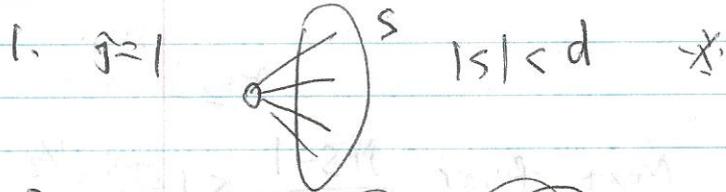
First $d/2$ factors

$$\leq \left(\frac{\lceil j+s \rceil^{2r/d}}{[n]^{2r/d}} \right)^{d/2} = B^{d/2}$$

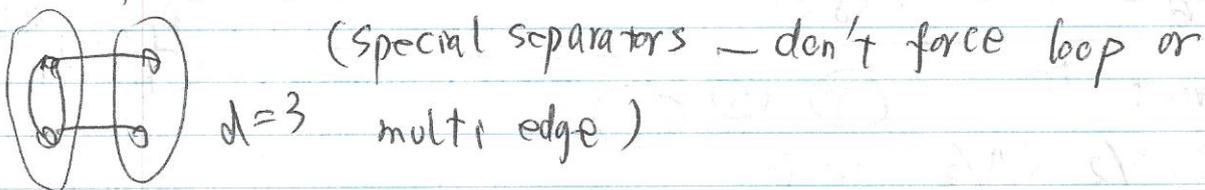
$$\frac{j+s - 1/d}{n - 1/d} \dots \frac{j+s - \frac{d-1}{d}}{n - \frac{d-1}{d}} \bigg/ \frac{j+s - \frac{d+1}{d}}{n - \frac{d+1}{d}}$$

all $\leq \frac{j+s}{n}$

Left with $j < 4s$



Prob = $O(\frac{1}{n})$ (Bounded number of cases for fixed d)



$j=2$ Finally consider $j \geq 3$ $j < 4s$ ($d \geq 3$)

Claim this gives $r > j+s$, so subgraph induced by T, S has more edges than vertices.

well, $r = \binom{d+s}{2} \geq \frac{d+s}{2} = j+s + \boxed{j(\frac{d}{2}-1) - \frac{s}{2}}$

CO739 July 15th, 2008 Sunny

Random Graph

a.a.s. d -connected ($d \geq 3$)

d -regular graph.

$$j(d-2) - s \quad (s \leq d-1)$$

$$\geq j(d-2) - (d-2) + 1$$

n even, perfect matchings



Thm. If G is d -regular

$$j(d-2) - 1 > 0$$

and $(d-1)$ -edge-connected, and $|V(G)|$ even, then G has a p.m.

Cor. for $d \geq 3$ n even, $G_{n,d}$ a.a.s. has a perfect matching.

$d=2$? Union of disjoint cycles



Has a p.m. iff all cycles are even.

X_i = # of i -cycles in pairing model.

$$\left(\frac{1}{2i}\right)$$

X_1, \dots, X_{2k} are a.s. indep. Poisson

$$\mathbb{P}(X_i) \rightarrow \frac{(d-1)^2}{2i}$$

$$G_{n,d} = \mathcal{P}_{n,d} \mid X_1 = X_2 = 0$$

$$\begin{aligned}
 & P(\text{no odd cycle of length } 1, \dots, 2k-1 \text{ in } \mathbb{P}_{n,d}) \\
 & \sim \prod_{i=1}^k e^{-\frac{1}{2i-1}} = e^{-\frac{1}{2} \sum_{i=1}^k \frac{1}{2i-1}} \\
 & = e^{-\frac{1}{2} (\ln(2k) + O(1)) - \frac{1}{2} \ln k + O(1)} \quad (\text{all}) \quad (\text{even}) \\
 & \sim e^{-\frac{1}{4} \ln k} \cdot \Theta
 \end{aligned}$$

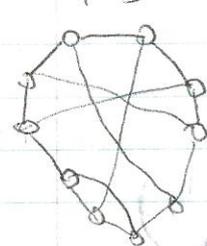
So in $\mathbb{G}_{n,d}$, prob is $\leq \frac{1}{P(\text{simple})} \cdot O(e^{-\frac{1}{4} \ln k}) < \varepsilon$ for k suff. large (any $\varepsilon > 0$)

\rightarrow const. \approx around $(\ln n)$

So qqs. $\mathbb{G}_{n,2}$ has an odd cycle & no perfect matching
 if $\mathbb{G}_{n,d}$ connected, then \Rightarrow p.m if n is even.

n odd? $P(\text{no odd cycles with length } < \ln n) < \varepsilon + o(1) \quad \forall \varepsilon > 0$
 $\therefore = o(1)$

\Rightarrow qqs. not connected.
 $d=3$ (cubic)



(deleting, p.m., still regular \therefore) does not have uniform 2-regular graph.

Let $\gamma =$ no. of ham. cycles in $\mathbb{P}_{n,d=3}$

$$\begin{aligned}
 E\gamma &= \frac{n!}{2^n} \cdot 6^n \cdot \frac{n!}{(n/2)! \cdot 2^{n/2}} \cdot \frac{(3n/2)! \cdot 2^{3n/2}}{(3n)!} \cdot \frac{(n-1)!!}{(3n-1)!!} \\
 & \sim \frac{\sqrt{2\pi n}}{2^n} \cdot 6^n \cdot \frac{\sqrt{2\pi n} \left(\frac{3}{2}\right)^{3n/2}}{\left(\frac{1}{2}\right)^{n/2} 2^{n/2}} \cdot \frac{\sqrt{2\pi \frac{3n}{2}} \cdot 2^{3n/2}}{\sqrt{2\pi \frac{3n}{2}} \cdot 2^{3n/2}} \cdot \frac{\left(\frac{n}{e}\right)^{n-1+3n/2-n/2-3n}}{\left(\frac{n}{e}\right)^{3n-1}} \\
 & = 1
 \end{aligned}$$

prob with a loop is const.

$$= \sqrt{\frac{\pi}{n}} \cdot \frac{\sqrt{2} \sqrt{3/2}}{2 \sqrt{1/2} \sqrt{3}} \left(\frac{6 \cdot (3/2)^{3/2}}{3^3} \right)^n = \sqrt{\frac{\pi}{2n}} \left(\frac{2}{\sqrt{3}} \right)^n$$

$$E(Y(Y-1)) = \frac{n!}{2n} X \dots \text{(1) (1st Ham. cycle)}$$

ordered pairs of distinct H -cycles.

How many matchings of k -edges does an n -cycle have?



Edges in H_1, H_2 are a matching.

$(n-2k)$ vertices "interior" of k segments.
 (edges appear in 1st H -cycle but not in the 2nd).

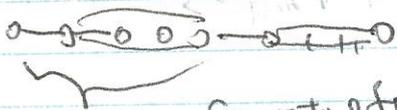
$$= f(n, k)$$

with distinguished matching edge is $k f(n, k)$

$$k f(n, k) = n [x^n] \cdot \left(\frac{x^2}{1-x} \right)^k$$

place dist. edge.

binary sequence prob.



Generating func.

$$x^2$$

for matching edge

$$\frac{x^2}{1-x}$$

+ "segment" of $H_1 \cap H_2$

$$x \frac{n}{k} \binom{n-k-1}{k-1} \quad (2)$$

[place of segments]

$$= n [x^{n-2k}] (1-x)^{-k}$$

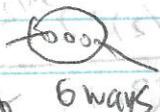
$$= n \binom{-k}{n-2k} = n \binom{n-k-1}{k-1}$$

$$x \frac{(2b-2)(2k-4) \dots (2)}{2^{b-1} (b-1)!} \quad (3)$$

[join segments into H_2]

$$x 6^n \dots \quad (4)$$

(wiring up pts)

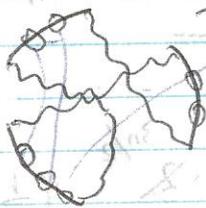
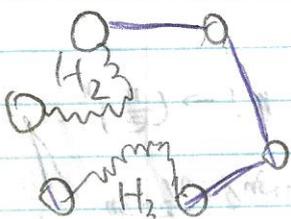


6 ways

$$x \frac{(n-2b-1)!}{(3n-1)!} \quad (5)$$

[wire up $n-2b$ remaining pts]

& divide by $\Phi_{n,d}$



$$\sum_{k=1}^{n/2} \frac{n!}{2^k k!} \frac{(n-k)!}{(k-1)! (n-2k)!} \cdot 2^{b-1} (k-1)! \cdot \frac{(n-2k)!}{(\frac{n}{2}-k)!} \cdot \frac{(3n)! \cdot 2^{3n}}{(3n)!} \cdot 2^{3n}$$

$$= \frac{n! 6^n (\frac{3n}{2})! \cdot 2^{3n/2 - n/2}}{(3n)!} \sum_{k=1}^{n/2} \frac{1}{k} \frac{(n-k-1)! \cdot 2^{2k}}{(\frac{n}{2}-k)!}$$

lemma: $\sum_{k=1}^{n/2} t_k \sim (\frac{8n}{3e})^{n/2} \cdot \frac{\sqrt{\pi}}{2n^{3/2}} t_k$ was $3/2$ (wrong)

So $E(Y(Y-1)) \sim (\frac{n}{e})^n \sqrt{2\pi n} \cdot 6^n (\frac{3n}{2e})^{3n/2} \sqrt{2\pi \frac{3n}{2}} \cdot 2^n \cdot X$

$$= \frac{\pi}{n} \cdot \left(\frac{6 \cdot 3^{3/2} \cdot 2 \cdot 8^{1/2}}{2^{3/2} \cdot 3^3 \cdot 3^{1/2}} \right)^n \cdot \frac{3}{8}$$

$$= \frac{\pi}{n} \left(\frac{4}{3} \right)^n \quad EY \sim \sqrt{\frac{\pi}{2n}} \left(\frac{2}{3} \right)^n$$

July 17th 2008 6.739 Random Graph

$$E(Y(Y-1)) \sim \frac{\pi}{n} \cdot \frac{3}{8} \left(\frac{4}{3} \right)^n$$

$$EY^2 \sim \sqrt{\frac{\pi}{2n}} \left(\frac{2}{3} \right)^n \rightarrow \infty$$

$$E(Y(Y-1)) \sim (EY)^2 \cdot \left(\frac{3}{4} \right) \text{ was } \frac{3}{2}$$

chebyshev $\frac{3}{2}$

$$E(Y(Y-1)) = EY^2 - EY$$

$$= E((Y-EY)^2) + (EY)^2 - EY$$

$\geq (EY)^2 - EY$ negligible

$$P(|Y-EY| \geq EY) \leq \frac{\sigma^2}{(EY)^2}$$

$$P(Y=0) \leq \frac{\sigma^2}{(EY)^2}$$

$$\sigma^2 = \text{var}(Y) = EY^2 - (EY)^2 = 2(EY)^2$$

so $P(Y=0) \leq 2 + o(1)$

Alternative second moment inequality no loops $\left[\begin{matrix} x=0 \\ e^{-\theta} \end{matrix} \right]$ $P_{n,3}$ $x \geq 1$ simple e^{-1} Hilroy

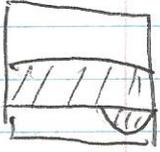
So here $P(X \neq 0) \geq \frac{1}{3}$

But P has a loop $\Rightarrow X(P) = 0$

$P(\text{no loop}) \sim e^{-(3-1)/2} = e^{-1}$

So $P(X \geq 1) \cdot P(\text{simple}) \sim e^{-2}$

In $G_{n,3}$ $\mathbb{P}(\text{no loop not simple}) \sim \frac{1}{e} - \frac{1}{e^2}$



so $\mathbb{P}(\text{simple} \wedge X=1) \geq \frac{1}{3} - (\frac{1}{e} - \frac{1}{e^2}) > 0$

so $\mathbb{P}_{G_{n,3}}(X=1) \geq \frac{\alpha}{e^{-2}} (\text{Hous}) = \frac{e^2}{3} - e + 1 > 0$

proof of alternative 2nd moment ineq.

$$\mathbb{R}E X = \mathbb{E}(X I_{X \neq 0}) = \sum_{y \in V} (P(y) X(y) I_{X \neq 0}(y))$$

$$= \sum_{y \in V} (\overline{I P(y)} \cdot X(y)) (\overline{I P(y)} I_{X \neq 0}(y))$$

$\frac{\langle \alpha, \beta \rangle}{\|\alpha\| \|\beta\|} \leq 1$ Cauchy-Schwarz $\leq \sqrt{\sum_y P(y) X(y)^2} \sqrt{\sum_y P(y) I_{X \neq 0}(y)}$

so $(\mathbb{E} X)^2 \leq \mathbb{E}(X(y)^2) P(X \neq 0)$ \square

proof of the Lemma: $t_k = \frac{1}{k} \frac{(n-k-1)!}{(\frac{n}{2}-k)!} \cdot 4^k$

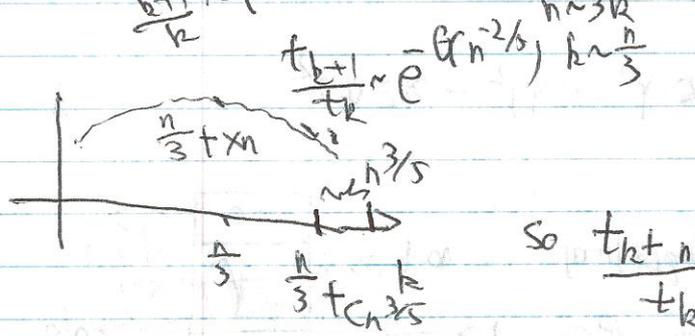
$\frac{t_{k+1}}{t_k} \sim \frac{\binom{n-k}{2}}{\binom{n}{2}-k} \cdot 4$

for k much larger ratio goes down.

for $(k > \log n)$ ratio ~ 1 (if $2n-4k \sim \frac{n}{2}-k$)

for $k = \frac{n}{3} + cn^{3/5}$

$$\frac{\frac{2n}{3} - 2cn^{3/5}}{\frac{n}{3} - cn^{3/5}} \times 2 = \frac{\frac{2n}{3} - 4cn^{3/5}}{\frac{n}{3} - cn^{3/5}}$$



$= 1 - \theta(n^{-2/5})$

so $\frac{t_{k+n^{3/5}}}{t_k} \sim e^{-\theta(n^{1/5})} < n^{-k} \forall k$ fixed

similar calculation for $k < \frac{n}{3} - \theta^{3/5}$

so put $S_k = k t_k$ then $\sum_{h=1}^{n/2} S_k = \left(\sum_{h=\frac{n}{3}-1}^{\frac{n}{3}+1} \right) (1 + O(n^{-1/2}))$.57

so similar for t_k since $k = O(n)$

takes stirling. Note $t_k = \frac{1}{k(n-k)} \frac{(n-k)!}{(\frac{n}{2}-k)!} 4^k$

$$t_k \sim \frac{9}{2n^2} \left(\frac{n-k}{e} \right)^{n-k} \frac{1}{\sqrt{2\pi \cdot \frac{2}{3}n}} \quad n-k \sim \frac{2}{3}n$$

$$\left(\frac{\frac{n}{2}-k}{e} \right)^{\frac{n}{2}-k} \sqrt{2\pi \cdot \frac{1}{6}} \quad \left[k = \frac{n}{3} + O(n^{3/5}) \right]$$

Put $k = n/3 + Xn$

$$t_k \sim \frac{9}{n^2} \left(\frac{n}{e} \right)^{n/2} \frac{2}{(\frac{2}{3}-X)^{\frac{2}{3}-X}} \frac{n-k}{e} = \frac{n - \frac{n}{3} - nX}{e} = \left(\frac{n}{e} \right) \left(1 - \frac{1}{3} - X \right)$$

$$\left(\frac{1}{6}-X \right)^{\frac{1}{6}-X} \frac{1}{4^{\frac{1}{3}+X}} = \dots$$

log f? $f =$

$$\log(a-x)^{a-x} = (a-x) \ln(a-x) = (a-x) \ln(a(1-\frac{x}{a}))$$

$$X = \frac{1}{3} - k \quad = (a-x) \ln a + (a-x) \ln \left(1 - \frac{x}{a} \right)$$

$$x = O(n^{-2/5}) \quad = (a-x) \ln a + (a-x) \left(-\frac{x}{a} - \frac{1}{2} \frac{x^2}{a^2} + O(n^{-1}) \right)$$

$$= (a-x) \ln a - x + \frac{x^2}{a} - \frac{1}{2} \frac{x^2}{a} + O(n^{-1}) \quad O(n^{-1})$$

$$\ln f = \frac{3}{2} \ln 2 - \frac{1}{2} \ln 3 - \frac{9}{4} X^2 + O(n^{-1})$$

$$\text{so } f^n = e^{n \ln f} = \frac{\sqrt{8}^n}{3^n} e^{-\frac{9}{4} X^2 n} o(1)$$

$$\text{so } \sum t_k \sim \frac{9}{n^2} \left(\frac{n}{e} \right)^{n/2} \left(\frac{8n}{3e} \right)^{n/2} \sum_{j=-n^{3/5}}^{n^{3/5}} e^{-\frac{9}{4} X^2 n}$$

$$j = k - n/3 \quad x = \frac{j}{n}$$

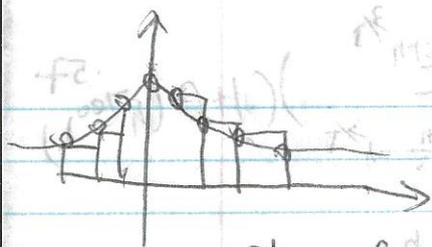
Note $\sum \sim \sum_{j=-\infty}^{\infty} e^{-\frac{9}{4} j^2/n}$

We are done if

$$\sum \sim \frac{2}{3} \sqrt{11n}$$

$$x^2 = \frac{j^2}{n^2}$$

j is integers
or ints + 1/3 or
ints + 2/3



$$|z - j| \leq 1$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt = 1$$

and $\int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}$

$$\int_{-\infty}^{\infty} e^{-t^2/a} dt = \sqrt{\pi a}$$

$$\text{So } \Sigma \sim \frac{2}{3} \sqrt{\pi n}$$

July 22nd 2008 CO739 Random Graph

3-regular $G_{n,3} = P_{n,3}$

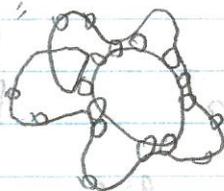
$Y = \# \text{ Ham cycles}$ $EY \sim \left(\frac{2}{3}\right)^n \sqrt{\frac{\pi}{2n}}$

$$EY^2 \sim 3(EY)^2$$

$$Y = [x_1]_{c_1} \dots [x_k]_{c_k}$$

$E(Y | \text{simple}) = \frac{\# \text{ Ham-cycles in "simple pairings"}}{\# \text{ simple pairings}}$

$EY^2 | \text{simple} = \# \text{ Ham-cycles in } P_{n,d}$



$$= \frac{\# (\text{Ham cycle } H, \text{ } i\text{-cycle } C)}{|P_{n,d}|} = \sum_{H,C} P(H,C \subseteq P_{n,d})$$

No remaining pts. = $n - 2k$

where the intersection $H \cap C$ forms k paths. ($k \geq 1$)

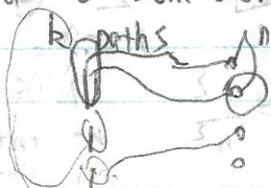
$$k \leq i/2$$

$$\text{So } P(H,C \subseteq P_{i,d}) = \frac{(n-2k-1)!!}{(3n-1)!!} \sim \frac{(i-1)!!}{(\frac{3}{2}i)!!} = \left(\frac{1}{e}\right)^{i/2}$$

$$\text{Prob} \sim \frac{\left(\frac{n-2k}{e}\right)^{\frac{n}{2}-k-k}}{\left(\frac{3n}{e}\right)^{3n/2}} \sim \frac{n^{-k}}{\left(\frac{n}{e}\right)^n 3^{3n/2}}$$

of C in pairing is $\frac{[n]_i}{2^i} \cdot i$ # ways to choose k "gaps" between $H \cap C$ paths $f(i,k)$

(Need to sum over $k \leq \lfloor i/2 \rfloor$ at the end)



$n-i+k$ vertices. $n-i$ buckets. $n-i+k$ vertices. Ham cycles on $n-i+k$ vertices. no automorphism.

$\sim n! n^{k-i}$

$(N_0 = \frac{(n-i+k)!}{2(n-i+k)} \times 2^k \text{ ways to hook onto the paths.}$

For pairs 6^{n-i} ways to wire up onto non-cycle buckets
 $EY X_i \sim \sum_{k=1}^{\lfloor i/2 \rfloor} \frac{n!}{2^i} \frac{n! n^{k-i}}{2^n} 2^k 6^{n-i} \frac{1}{n^k (\frac{n}{e})^n 3^{3n/2}} f(i, k)$

$= \frac{2^n}{3^{n/2}} \frac{\sqrt{2\pi n - 2k}}{4n^i} \sum_{k=1}^{i/2} 2^k f(i, k) \quad EY \sim (\frac{2}{3})^n \sqrt{\frac{n}{2}}$

$\Rightarrow EY X_i \sim EY \times \frac{1}{2^i} \sum_{k=1}^{\lfloor i/2 \rfloor} f(i, k) 2^k$

$f(i, k) = \frac{i!}{k} \binom{i-k-1}{k-1} = \binom{i-k}{k} \frac{k}{i-k}$

So $EY X_i \sim EY \cdot \frac{1}{2^i} \sum_{k=1}^{i/2} 2^k f(i, k) = EY \cdot \sum_{k=1}^{i/2} \frac{1}{2^k} \frac{k}{i-k} \binom{i-k}{k}$

$= EY \cdot \sum_{k=1}^{\lfloor i/2 \rfloor} \frac{1}{2} \left(\frac{1}{2}\right)^{i-k} \binom{i-k}{k} \frac{1}{i-k}$

$= EY \cdot 2^i \sum_{k=1}^{\lfloor i/2 \rfloor} \frac{1}{2} \frac{1}{i-k} [X^{i-k}] \left(\frac{X(1+X)}{2}\right)^{i-k}$

$= EY \cdot 2^i \cdot \frac{1}{2} [X^1] \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{X(1+X)}{2}\right)^j \quad (*)$

$i-k=1$
 $= EY \cdot 2^i \cdot \frac{1}{2} [X^1] \left(-\log\left(1 - \frac{X(1+X)}{2}\right)\right)$

$(-\log(1-u) = \sum \frac{1}{i} u^i) \quad 1 - \frac{X(1+X)}{2} = \frac{2-X-X^2}{2} = \frac{(2+X)(1-X)}{2}$

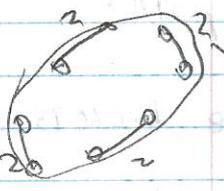
$= EY \frac{2^i}{2} \cdot \left(\frac{1}{2} \left(-\frac{1}{2}\right)^i + \frac{1}{2}\right) = (1 + \frac{X}{2})(1-X)$

$EY \cdot \frac{1}{2^i} + \dots = EY \left(\frac{2^i}{2^i} + \frac{(-1)^i}{2^i} - \frac{1}{2^i}\right) = \mu_i \cdot EY$

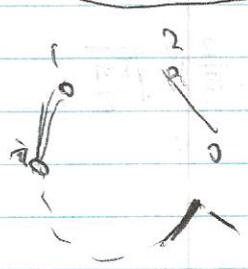
So $EY X_i \sim (EY) \mu_i$

$\mu_i = \frac{(-1)^i}{2^i} - \frac{1}{2^i} + \lambda_i$

other methods to evaluate $\sum_k 2^k f(n, k)$



gen fun. $\frac{2x}{1-x}$
 weight $[x^i] \frac{1}{1-2x}$



in not
 $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

A^i gives no. sequences of length i with 2 $\#(not \rightarrow in)$

in not $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $\text{trace}(A^i) = \sum f(n, k) 2^k = \sum \text{e-vals}(A)^i$

$\det A = (1-x)(-x) - 2 = x^2 - x - 2 = (x-2)(x+1)$

evals 2, -1,

$\therefore 2^i + (-1)^i$ walk "in" always

Change prob to make

$P(p) \sim Y$ so $P_{new}(p) = \frac{Y(p)}{\sum_{p'} Y(p')}$

$E_{new} X_i = \sum_p X_i(p) \frac{Y(p)}{\sum_{p'} Y(p')} \leftarrow E_{P_{new}} Y^X | P_{new}$

$= \sum_p \frac{X_i(p) \cdot Y(p)}{E_{old} Y} \times \frac{1}{|P_{new}|} = \frac{1}{E_{old} Y} E_{old} Y X_i'$

$\sim M_i'$

similar calc give

$$\mathbb{E} Y [X_1]_{i_1} \dots [X_r]_{i_r} \sim (\mathbb{E} Y)^{i_1} \dots \mu_j^{i_j}$$

so short cycles as indep Poisson with mean = $\mu_j^{i_j}$ in
"ham-weighted" model.

July 24th 0739 Random Graph Prof. Nick Wormald

$$\frac{\mathbb{E} Y^2}{(\mathbb{E} Y)^2} \rightarrow 3 \quad P_{n,d} \quad \lambda_i = \mathbb{E} X_i + \alpha(i) \quad \text{i-cycles}$$

$$\frac{\mathbb{E} Y [X_1]_{i_1} \dots [X_k]_{i_k}}{(\mathbb{E} Y)^{i_1 \dots i_k}} \rightarrow \mu_j = \frac{2^i + (-1)^i - 1}{2^i}$$

Thm: let $X_1, \dots, X_k \geq 0$ r.v.s st. X_1, \dots, X_k as indep Poisson $\forall k$.
suppose $Y \geq 0$ r.v. with $\mathbb{E} Y > 0$ n suff large. and

$$(a) \frac{\mathbb{E} Y [X_1]_{i_1} \dots [X_k]_{i_k}}{(\mathbb{E} Y)^{i_1 \dots i_k}} \rightarrow [\mu_j]_{i_j} \quad \forall \text{ fixed } i_1, \dots, i_k, k$$

let with $\mu_j = \lambda_j (1 + \delta_j)$ (defn) $[\dots \delta_j \geq -1]$ (fixed $\mu_j \geq 0$)
and

$$(b) \frac{\mathbb{E} Y^2}{(\mathbb{E} Y)^2} \rightarrow \exp\left(\sum_{i=1}^{\infty} \lambda_i \delta_i^2\right) \text{ \& } \sum \text{ converges}$$

Then. $\forall \epsilon > 0 \exists \epsilon_1 > 0 \mathbb{P}\left(\frac{Y}{\mathbb{E} Y} < \epsilon_1\right) < \epsilon$ [Cor. $\epsilon > 0$ a.s. $|X_i| \leq \epsilon$]

Application: $Y =$ Ham cycles in $P_{n,3}$ model etc

$X_i = \#$ i -cycles.

$$\mu_j = \frac{2^j + (-1)^j - 1}{2^j} = \lambda_j \left(1 + \frac{(-1)^j - 1}{2^j}\right) \Rightarrow \delta_j = \frac{(-1)^j - 1}{2^j}$$

$$\delta_j^2 = -2 \frac{(-1)^j - 1}{2^j}$$

$$\sum_{j \geq 1} \lambda_j \delta_j^2 = \sum_{j \geq 1} \frac{-2}{2^j} \cdot \frac{(-1)^j - 1}{2^j} = \sum_{j \geq 1} \frac{1}{2^j} \left(\frac{1}{2}\right)^j - \frac{1}{2^j} \left(-\frac{1}{2}\right)^j$$

$$= -\ln\left(1 - \frac{1}{2}\right) + \ln\left(1 + \frac{1}{2}\right) = \ln 3$$

$$\mathbb{P}\left(\frac{Y}{\mathbb{E} Y} < \epsilon_1 \mid X_i = 0 \forall i \in A\right) < \epsilon \quad \text{where}$$

$$A = \{i : \delta_i = -1\}$$

when is $\delta_j = -1$? $j=1$.

$$\begin{aligned}
 & -1 + \sum_{i \geq 1} \frac{i}{i!} e^{\lambda - 2\mu} \left(\frac{\mu^2}{x}\right)^i \\
 & = -1 + 0(1) + e^{\lambda - 2\mu + \mu^2/\lambda} \\
 & = -1 + 0(1) + e^{\lambda(1 - 2 - 2\delta + (1+\delta)^2)} \\
 & = -1 + 0(1) + e^{\lambda\delta^2}
 \end{aligned}$$

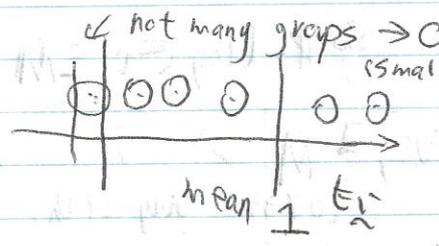
$\mu = \lambda(1+\delta)$

Define $\tilde{r} = (i_1, \dots, i_k)$, group means $E_{\tilde{r}} = (E(Y | X_1=i_1, \dots, X_k=i_k))$
 variances V_i .

$\sum \lambda_j \delta_j^2$
 from part (b) find C as above

$$E Y^2 - (E Y)^2 = \sum_i E V_i + \text{Var } E_{\tilde{r}}$$

random thing



$$\text{Var } E_{\tilde{r}} \geq -1 + 0(1) + e^{\sum_{j=1}^k \lambda_j \delta_j^2} \quad (\text{sums of products})$$

(b) $\Rightarrow \forall \epsilon > 0 \exists k \quad E V_i = \epsilon$

Restrict \tilde{r} . $A = \{j, \delta_j = -1\}$

Forget about these indices. i.e. assume all $\delta_j > -1$

let $I_0 = \{j : \delta_j < -\frac{1}{2}\}$

$\sum_{j \in I_0} \lambda_j \delta_j^2$ bounded $\sum_{j \in I_0} \lambda_j$ is bounded

$$E_{\tilde{r}} \text{ was } \sim \prod_{j=1}^k e^{\lambda_j - \mu_j} \left(\frac{\mu_j}{x_j}\right)^{i_j} = \prod_{j=1}^k e^{-\lambda_j \delta_j} (1+\delta_j)^{i_j}$$

so $\exists N_0 : \sum_{j \in I_0, j > N_0} \lambda_j < \epsilon$ N_0 indep of k .

let $I_1 = \{j : i_j = 0 \text{ for } j \in I_0 \text{ \& } j > N_0\}$

$$IP((x_1, \dots, x_k) \notin I_1) \leq \sum_{\text{such } j} \lambda_j = O(\epsilon)$$

fix ϵ , $\{j : j \leq N_0\}$ is bounded.

$-\frac{1}{2} \leq \delta_j \leq -\frac{1}{2}$ All other $\delta_j > -\frac{1}{2}$

$$\log(E_{\tilde{X}}) = \sum_{j \geq 1} -\lambda_j \delta_j + \sum_j \log(1 + \delta_j)$$

$$E_{\tilde{X}} = \sum_{j \geq 1} -\lambda_j \delta_j + \sum_j (\delta_j + O(\delta_j^2)) \text{ for such } \tilde{X}$$

So $E \log E_{\tilde{X}} \geq \sum_j \lambda_j O(\delta_j^2)$ which is bounded.

random
(X_1, \dots, X_k)

$$\text{So } P(\log E_{\tilde{X}} \leq -M) < \epsilon$$

($\forall \epsilon, \exists M \uparrow$) so $\forall \epsilon, \exists M, P(\log E_{\tilde{X}} < -M) < \epsilon$
 0739 July 29th, 2008 Random Graph

Thm. let $X_1, \dots \geq 0$ int. r.v.'s

$E X_i \rightarrow \lambda_i > 0$ asymp. indept Poisson (λ_i)
 $Y_n > 0$ (n large) and

(a) $E \prod_{i=1}^k [X_i]_{i_k} \rightarrow \prod_{i=1}^k \lambda_i^{i_k} \quad \forall i_1, \dots, i_k \text{ fixed.}$
 ($\forall k$)

(b) let $\delta_j = \frac{\mu_j}{\lambda_j} - 1 \quad \mu_j = (1 + \delta_j) \lambda_j$

$$\frac{E Y^2}{(E Y)^2} \sim e^{\sum \lambda_j \delta_j^2} \quad (\sum \text{ converges })$$

Then. Define $A := \{j : \delta_j = -1\}$; assume $|A|$ finite.

proof: $\forall \delta > 0 \quad \exists \epsilon > 0 \quad P(\frac{Y}{E Y} \geq \delta) \geq 1 - \delta \quad |X_j = 0 \quad \forall j \in A$
 1st assume $A = \emptyset$ (assume = 1) condition

$$E_{\tilde{X}} = E(Y | \underline{X} = \underline{v}) \quad \underline{X} = (X_1, \dots, X_k) \text{ - random vector.}$$

$$V_{\tilde{X}} = \text{var}(Y | \underline{X} = \underline{v}) \quad \text{showed: (i) } \forall \delta > 0, \exists k = k(\delta)$$

$$E V_{\tilde{X}} < \zeta + o(1) \quad \sum p_{\tilde{X}} V_{\tilde{X}}$$

(2) $\forall \epsilon > 0 \quad \exists \epsilon_1 \quad P(E_{\tilde{X}} < \epsilon_1) < \epsilon$
 \tilde{X} is indept of k .

To show conclusion. Given any $\epsilon > 0$, choose ϵ_1 as above

$$\text{Now let } \zeta = \epsilon^2 \epsilon_1^2$$

Then by (1) & Markov

$$IP(V_{\underline{x}} \geq \varepsilon \varepsilon_1^2) \leq \frac{EV_{\underline{x}}}{\varepsilon \varepsilon_1^2} \leq \frac{\varepsilon^2 \varepsilon_1^2}{\varepsilon \varepsilon_1^2} + o\left(\frac{1}{\varepsilon \varepsilon_1^2}\right) \rightarrow 0$$

Let $I_2 = \{i : V_{\underline{x}} < \varepsilon \varepsilon_1^2\} = O(\varepsilon)$

Now let $\underline{x} \in I_2$ & $E_i \geq \varepsilon_1$ Then $IP(\underline{x} \in I_2) \geq 1 - O(\varepsilon)$

$$IP(Y < \frac{E_i}{2} | \underline{x} = i) \leq IP(|Y - E(Y|\underline{x}=i)| > \frac{E(Y|\underline{x}=i)}{2})$$

$$E_i = E(Y|\underline{x}=i) \leq \frac{V_i}{(E_i/2)^2} \leq \frac{\varepsilon \varepsilon_1^2}{(\varepsilon_1/2)^2} = O(\varepsilon)$$

so $IP(Y < \frac{\varepsilon_1}{2}) = O(\varepsilon) \leq M\varepsilon$ (some absolute M)

So given δ , put $\varepsilon = \frac{\delta}{M}$, then \forall

$$IP(Y \geq \frac{\varepsilon_1}{2}) \geq 1 - \delta$$

for $G_{n,d}$ in general, same method gives \exists Ham cycle a.s. if $d \geq 3$ (fixed d) [bigger computation]

4-regular (n even) a.s. has perfect matching

4-regular - pm = 3-regular.

$G_{n,4} - G_{n,1}$

not get uniform dist.

induction (delete ... problem, add ... usual way)

or 3-regular + random perfect matching \rightarrow 4-regular simple.

let $Y =$ no. perfect matchings in $G_{n,d}$.

$X_j = \#$ j-cycles. $\lambda_0 = \frac{(d-1)^2}{2}$ as before

— Compute $\rightarrow \mu_i = \lambda_i(1+d_i)$ where $\delta_i = \left(\frac{1}{d-1}\right)^i$

$$\frac{EY^2}{(EY)^2} \rightarrow \sqrt{d-1/d-2}$$

exercise. "paper"

easy to check $\sum_{j \geq 1} \lambda_j \delta_j^2 = \sum \frac{(d-1)^2}{2^j} \left(\frac{1}{d-1}\right)^{2j} = \frac{1}{2} \sum \frac{1}{2^j} \left(\frac{1}{d-1}\right)^{2j}$ Random Graphs Hilroy

$$= \frac{1}{2} \left(-\log\left(1 - \frac{1}{d-1}\right)\right) = -\frac{1}{2} \log \frac{d-2}{d-1}$$

$$e^{\alpha} = \sqrt{\frac{d-1}{d-2}} \quad \forall \delta > 0, \exists \epsilon > 0: \text{st.}$$

$$P(Y/\epsilon Y \geq \epsilon) \geq 1 - \delta + o(1)$$

(eg \Rightarrow gas \square p.m)

Superposition:

Let $\mathcal{G}, \tilde{\mathcal{G}}$ be graph prob. spaces, same vertex set.

$\mathcal{G} + \tilde{\mathcal{G}}$ is random graph obtained from taking $G \cup \tilde{G}$ where $G \in \mathcal{G}, \tilde{G} \in \tilde{\mathcal{G}}$

suppose we reject if mul. edge occurs.

$$\rightarrow \mathcal{G} \oplus \tilde{\mathcal{G}} = \mathcal{G} + \tilde{\mathcal{G}} \Big|_{\text{simple}} \quad [\text{makes sense if } P(\text{simple}) > 0]$$

$$\mathcal{G}^{(Y)} \quad Y \geq 0, \epsilon Y > 0$$

\sim P is multiplied by $\frac{Y}{\epsilon Y}$

Let S_n be a set of k -regular graphs on $V = [n]$.

Let $\mathcal{G} = \#$ of subgraphs of G in S_n .

Then in $\mathcal{G}_{n,d}^{(Y)}, P(G)$

marginal distribution of G in $\{(G, H) : H \subseteq G, H \in S_n\}$ with

uniform distribution. $\hookrightarrow \{(F \cup H, H) : F \cup H \in \mathcal{G}_{n,d}, H \in S_n, E(F) \cap E(H) = \emptyset\}$

marginal of $F \cup H$ in

= marginal of \dots = dist. of $F \cup H$ in $\{(F, H), \dots\}$

If S_n has uniform distribution.

$$F \in \mathcal{G}_{n, d-k}, H \in S_n, E(F) \cap E(H) = \emptyset$$

this is exactly $\mathcal{G}_{n, d-k} \oplus \mathcal{G}_{n, k}$.

graph-restricted sum.

$$\text{eg. } Y = \# \text{ of pms in } \mathcal{G}_{n,d} \quad \mathcal{G}_{n, d-1} \oplus \mathcal{G}_{n, 1} = \mathcal{G}_{n,d}^{(Y)}$$

new definition: let $\mathcal{G}_n (n \geq 1), \tilde{\mathcal{G}}_n$ be two sequences of prob. spaces.

with same underlying sets:

$G_n \approx \tilde{G}_n$ ("contiguous") if \forall events A (A_n depends on n)
($G_n \in A$ aas) \iff ($\tilde{G}_n \in A$, aas)

Theorem: $G^{(r)} \approx G$ ($r_n \geq 0, \exists r_n > 0$ n suff. large)

iff. $\forall \epsilon > 0 \exists \delta > 0$

$P(\delta < \frac{r}{\epsilon Y} < \frac{1}{\delta}) \geq 1 - \epsilon$

Note $P(\frac{r}{\epsilon Y} > \frac{1}{\delta}) < \delta$ by Markov

Small subgraph conditioning before

Gr's. If γ is # Ham cyc in $G_{n,3}$ (or $G_{n,d}$ $d \geq 3$)

or # P_m in $G_{n,d}$ ($d \geq 3$)

Then $G_{n,d} \approx G_{n,d}$

$\gamma = p, m$

$G_{n,d-1} \oplus G_{n,1} \approx G_{n,d}$

(throw multiple edges)

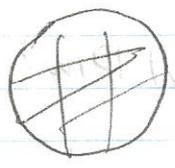
$\Rightarrow G_{n,4}$ aas Ham.

aas. Ham for $d=4$

Iterate $\rightarrow G_{n,d}$ aas Ham $\forall d \geq 3$ (fixed)

proof of Ham in $G_{n,3}$ showed $G_{n,1} \oplus H_n \approx G_{n,3}$

more general, $G_{n,d-2} \oplus H_n \approx G_{n,d}$ (its already Ham.)



small world.

problem $H_n + P_m$

diameter aas $\log_{(d-1)} n + \log \log \dots$

range of 6 integers.

(0739 Random Graph July 31, 2008)

$\Omega_n \approx \hat{\Omega}_n$ if $\forall A_n, P(A_n) \rightarrow 1 \iff P_{\hat{\Omega}_n}(A_n) \rightarrow 1$

$G_{n,d} \approx G_{n,d}$ $d \geq 3$

$P_n(A_n) \geq \frac{1}{2}?$

Lemma: $\forall \epsilon > 0 \exists \delta P(\delta \leq \frac{r}{\epsilon Y} \leq \frac{1}{\delta}) \geq 1 - \epsilon + o(1)$

iff $\Omega \approx \hat{\Omega}$ $Y = Y_n \rightarrow 0$ for n suff. large

wlog. $EY = 1$ ($Z = Y/EY$)

proof: Suppose $\Omega \approx \Omega^{(r)} \Rightarrow \forall \epsilon > 0 \dots$

(a) By Markov, $P(Y > \frac{1}{\delta}) < \delta$ put $\delta = \frac{\epsilon}{2}$

$$P(Y > \frac{1}{\delta}) < \frac{\epsilon}{2}$$

(b) suppose $\nexists \delta > 0$ $P(Y_n < \delta) < \frac{\epsilon}{2} + o(1)$

Then \exists arbitrarily small δ_n 's with $P(Y_n < \delta_n) \geq \frac{\epsilon}{4}$ (*)

(if $\exists \delta > 0$ with $P(Y_n < \delta) < \frac{\epsilon}{4}$ for all n (special n)

$$< \frac{\epsilon}{2} + o(1)$$

ie. (*) $\Rightarrow \exists \delta_n \rightarrow 0$ [same δ_n for special n
 $\rightarrow 0$ for non special.]

$$P(Y_n < \delta_n) \geq \frac{\epsilon}{4} \text{ infinitely often.}$$

$$\rightarrow P(Y_n < \delta_n) \not\rightarrow 0 \text{ (prob in } \mathbb{R}^1)$$

$$\text{But } P_{\Omega^{(r)}}(Y_n < \delta_n) = \sum_{w \in \Omega} Y_n(w) P(w) \leq \sum \delta_n P(w) \leq \delta_n \rightarrow 0$$

So $P_{\Omega^{(r)}}$, event $Y_n < \delta_n$ is a.s. false.

- contradicts $\Omega \approx \Omega^{(r)}$

(ii) $\forall \epsilon \dots \Rightarrow \approx$ fixed

(a) Suppose $P_{\Omega}(A) = o(1)$ Let $M > 0$

$$P_{\Omega}(Y)(A) = \sum_{\substack{w: Y(w) \leq M \\ w \in A}} Y(w) \cdot P(w) + \sum_{\substack{w: Y(w) > M \\ w \in A}} Y(w) \cdot P(w)$$

$$\leq M P(A) = o(1) +$$

second sum: $\leq \sum_{w: Y(w) > M} Y(w) P(w)$

Note $EY = \sum_w Y(w) P(w) = \sum_{k \geq 0} \left(\sum_{w: k \leq Y \leq k+1} Y(w) P(w) \right)$

So second term $< \delta$ where $\delta \rightarrow 0$ if $M \rightarrow \infty$

so $P_{\Omega(r)}(A) = o(1)$
 (b) if $P(B) \rightarrow 0$ in $\Omega(r)$

then Let $\epsilon > 0$ find δ st. $P(\frac{\delta}{2} \leq r < \frac{1}{\delta}) \geq 1 - \epsilon + o(1)$

$$P_{\Omega}(B) = \sum_{\substack{w: r(w) < \delta \\ w \in B}} P(w) + \sum_{\substack{w: r(w) > \delta \\ w \in B}} P(w)$$

$$\leq \epsilon + o(1) + \frac{1}{\delta} P_{\Omega(r)}(B) = o(1)$$

$$\leq \epsilon + o(1)$$

so $P_{\Omega(r)}(B) = o(1)$

Fact 1: $G_{n,d-2} \oplus \text{Ham cycle} \approx G_{n,d}$ ($d \geq 3$)

Fact 2: $G_{n,d-1} \oplus G_{n,1} \approx G_{n,d}$ ($d \geq 3$) [n , even only]

Cor: $G_{n,4}$ aas. Ham. $G_{n,3} \oplus G_{n,1} \approx G_{n,4}$

$$(G_{n,3} \oplus G_{n,1}) \oplus G_{n,1} = x_1 \oplus (x_2 \oplus x_3)$$

\oplus is associative & commutative.

$n\Omega = \Omega \oplus \Omega \dots \oplus \Omega$ n times.

$$G_{n,3}^{(r)} = 3 G_{n,1}$$

$3 G_{n,1} ? \rightarrow r = \#$ of decompositions into 3 pm's.

Lemma: if $\Omega \approx \hat{\Omega}$ and $\Omega' \approx \hat{\Omega}'$ ($G \approx \hat{G}$ and $G' \approx \hat{G}'$) all same underlying set of same vertex set. given n)

then $G \oplus G' \approx \hat{G} \oplus \hat{G}'$ provided max degree bounded and probabilities are preserved (multiple edges occur not too much to avoid aas. events killed) if vertices relabelled in Ω

Fact 3: $3 G_{n,1} \approx G_{n,3}$ by small graph conditioning (SSC)

Fact 4: $G_{n,d-2} \oplus G_{n,2} \approx G_{n,d}$ by SSC ($d \geq 3$) Hilroy

Fact 5: $2H_n \cong G_{n,4}$ (SSC) $\gamma = F$ decompositions into 2 Ham cycles.

Theorem: For $d \geq 3$ fix j, k_i st.

$2j + \sum_{i \geq 1} k_i = d$ k_i corresponds to $G_{n,i}$

Then $G_{n,d} \cong jH_n \oplus \bigoplus_{i \geq 1} k_i G_{n,i}$

proof: By induction, on d . $i \geq 1$

induction step $j \geq 1$

$G_{n,d-2} \cong (j-1)H_n \oplus (\text{rest}) \oplus H_n$ & by lemma

provided $d-2 \geq 3$ ie $d \geq 5$

by fact 1 $\cong G_{n,d}$

if $j \geq 1$ & $d = 4$ or 3 , we need to show

$G_{n,4} \cong 2H_n$ (Fact 5)

or $G_{n,4} \cong H_n \oplus G_{n,2}$ (Fact 1)

or $G_{n,4} \cong H_n \oplus 2G_{n,1}$?

Have $G_{n,3} \cong G_{n,2} \oplus G_{n,1}$ (Fact 2)

$G_{n,1} \oplus G_{n,1}$ use lemma & Fact 2.

left with $j=0$ (exercise)

$G_{n,1} \oplus G_{n,1} \not\cong G_{n,2}$

End of this course \square